EE263 Final Exam, August 2022

• This is an 8-hour take-home exam. Please turn it in on Gradescope.
  Be aware that you must turn it in within 8 hours of downloading it. After that, Gradescope will
  not let you turn it in and we cannot accept it.

• You may use any books, notes, or computer programs, including searching online. You may not
discuss the exam or course material with others, or work in a group.

• Do not discuss the exam until 8/14, after everyone has taken it.

• If you have a question, please email the staff. We have done our best to make the exam unam-
  biguous. So unless there is a mistake, we are unlikely to say much.

• Please check your email during the exam, in case we need to send a clarification or announcement.

• We expect your solutions to be legible, neat, and clear. Do not hand in your rough notes. Please
try to simplify your solutions as much as you can. We will deduct points from solutions that are
  technically correct, but much more complicated than they need to be.

• Start each solution on a new page. Individual parts can be on same page.

• When a problem involves some computation, we expect a clear discussion and justification of
  what you did as well as the final numerical result.

• Because this is an exam, you must turn in your code. Include the code in your pdf submission.
  We will deduct points for missing code.

• When explaining the mathematical approach, you may not refer to Julia operators (e.g., the back-
  slash operator). You may, of course, refer to inverses of matrices, or other standard mathematical
  constructs.

• The data files for these problems can be found at the URL

  http://ee263.stanford.edu/goodbye_summer22.html

• Good luck!
1. Fitting piecewise polynomials. In class we discussed fitting a polynomial to data. Here, we consider the slightly more sophisticated problem of fitting a piecewise polynomial.

A function \( \hat{g} : \mathbb{R} \to \mathbb{R} \) is a \((k\text{-piece})\) piecewise polynomial with breaks \( \tau_1, \ldots, \tau_{k-1} \in \mathbb{R} \) if there exist polynomials \( p_1, \ldots, p_k : \mathbb{R} \to \mathbb{R} \) so that

- \( \hat{g}(s) = p_1(s) \) when \( s < \tau_1 \),
- \( \hat{g}(s) = p_i(s) \) when \( s \in [\tau_{i-1}, \tau_i) \), and
- \( \hat{g}(s) = p_k(s) \) when \( s \geq \tau_{k-1} \).

We say that \( \hat{g} \) has order \( d + 1 \) if each piece \( p_i \) has degree no greater than \( d \). In other words, for \( i = 1, \ldots, k \), there exists \( d + 1 \) coefficients \( x_{i,1}, x_{i,2}, \ldots, x_{i,(d+1)} \in \mathbb{R} \) so that

\[
p_i(t) = x_{i,1} + x_{i,2}t + x_{i,3}t^2 + \cdots x_{i,(d+1)}t^d.
\]

Given a dataset \( s_1, s_2, \ldots, s_m \in \mathbb{R} \) and \( g_1, g_2, \ldots, g_m \in \mathbb{R} \), and given specified breaks \( \tau_1, \ldots, \tau_{k-1} \), we are interested in finding a piecewise-polynomial predictor \( \hat{g} : \mathbb{R} \to \mathbb{R} \) of order \( d + 1 \) to minimize

\[
J = \sum_{i=1}^{n} (\hat{g}(s_i) - g_i)^2.
\]

a) Explain how to find \( \hat{g} \) to minimize \( J \). Explain when the optimal \( \hat{g} \) is unique.

b) The file `piecepoly.json` contains \( s \in \mathbb{R}^{100} \), \( g \in \mathbb{R}^{100} \), \( \tau \in \mathbb{R}^3 \) (called `tau`), and \( d = 3 \).

Find \( \hat{g} \) and plot it on the interval \([0, 4]\), on top of a scatter plot of the points.

Report the optimal coefficients and optimal cost.

c) Notice that \( \hat{g} \) found in (b) is not continuous. For example, \( p_1(1) \neq p_2(1) \).

Explain how to find \( \hat{g} \) to minimize \( J \) subject to the constraint that \( \hat{g} \) is continuous. In other words,

\[
p_i(\tau_i) = p_{i+1}(\tau_i) \text{ for } i = 1, \ldots, k - 1.
\]

Explain when the optimal \( \hat{g} \) is unique.

d) Using the same `piecepoly.json` data, find a continuous piecewise polynomial predictor \( \hat{g} \) to minimize \( J \). Plot it on top of the plot in part (b). Report the optimal cost.

(Aside: If, \( \hat{g} \) and its first \( d - 1 \) derivatives are continuous, we call \( \hat{g} \) a spline with knots at \( \tau_1, \ldots, \tau_{k-1} \). If \( d = 3 \), \( \hat{g} \) is called a cubic spline. We omit this exercise on the exam.)

e) Is the cost you found in (d) larger or smaller than in (b)? Will this always be the case?
2. Drone delivery. We want a drone to visit \(m\) positions \(p_1, \ldots, p_m \in \mathbb{R}^2\) at times \(s_1, \ldots, s_m \in \mathbb{R}_+\). Define the matrix \(P \in \mathbb{R}^{2 \times m}\) to have columns \(p_j\), where \(j = 1, \ldots, m\). The following Julia code (copy and paste works) defines these:

\[
P = [-1 - .5 0 0.5 1 0; 2 0 1 0 2 0]; \quad s = [3; 7; 10; 13; 17; 20];
\]

a) The dynamics of the drone are

\[
\ddot{q} = u,
\]

where \(q : \mathbb{R} \to \mathbb{R}^2\) is the position of the drone as a function of time, and \(u : \mathbb{R} \to \mathbb{R}^2\) is an input force as a function of time. Write this as a linear dynamical system of the form

\[
\dot{x} = Ax + Bu, \quad y = Cx,
\]

where \(y : \mathbb{R} \to \mathbb{R}^4\) is the position and velocity of the drone. In particular, define \(y\) to satisfy

\[
y_1 = q_1, \quad y_2 = q_2, \quad y_3 = \dot{q}_1, \quad y_4 = \dot{q}_2.
\]

Is the system stable? Is the system controllable?

b) We will discretize this system with a sample interval \(h > 0\). Assume the force is piecewise constant on sample intervals. Construct the exact discretization

\[
x_d(k + 1) = A_dx_d(k) + B_d u_d(k), \quad y_d(k) = C_d x_d(k),
\]

where \(x_d : \mathbb{Z}_+ \to \mathbb{R}^n\) satisfies \(x_d(k) = x(kh)\), and likewise for \(y_d\) and \(u_d\). Give \(A_d\), \(B_d\) and \(C_d\).

c) We operate the drone on the time interval \([0, T]\) where \(T = s_m\). Define \(n = T/h\). Assume the drone starts at the origin with velocity zero. Explain how to choose forces to steer the drone through the desired positions at the desired times, i.e. \(y_{1,2}(s_i) = p_i\), while minimizing

\[
J_1 = \sum_{k=0}^{n-1} \|u_d(k)\|^2_2.
\]

The drone need not be stationary when passing through the points \(p_1, \ldots, p_m\).

d) With \(h = 0.1\), use your method to compute the optimal \(u_d\). Report the optimal value of \(J_1\) that you obtained. Plot the components of \(u\) with respect to time. Plot the trajectory of the drone with axes \(q_1\) and \(q_2\), so that the plot shows the path followed by the drone. Mark on your plot the points \(p_i\) where the drone has deliveries.

e) Suppose we want to find forces minimizing \(J_1\) so that the drone is also stationary (velocity zero) when it is at position \(i\), in order to make a clean drop. Explain how to do this. Report the optimal cost. Plot the trajectory and control inputs as before. (Even if you cannot solve this part, you may complete (f) without the condition that the drone is stationary.)

f) Suppose that, in addition to being stationary, we want to avoid the drone jerking too much, in order to avoid damaging its payload. We penalize the discrete jerk. Define

\[
J_2 = \sum_{k=1}^{n-1} \|u_d(k) - u_d(k - 1)\|^2_2.
\]

Explain how to find forces to steer the drone to be at the positions \(p_1, \ldots, p_m\) with velocity zero at the desired times \(s_1, \ldots, s_m\), while minimizing \(J_1 + \mu J_2\), where \(\mu > 0\) is given.

For \(\mu = 100\), report \(J_1\) and \(J_2\). Plot the trajectory and controls as before. Make a trade-off curve with \(J_2\) on the horizontal axis, and \(J_1\) on the vertical. Briefly interpret the endpoints.
3. Computing with subspaces. In this question we compute $S \cap T$ for two subspaces $S, T \subset \mathbb{R}^n$. Parts (a), (b), and (e) are independent, and may be done even if you can not answer any other parts.

a) Suppose $S, T \subset \mathbb{R}^n$ are subspaces. Show that $S \cap T$ is a subspace.

b) Let $A \in \mathbb{R}^{n \times k_1}$ and $B \in \mathbb{R}^{n \times k_2}$. Give a matrix $C$ in terms of $A$ and $B$, satisfying
\[
\text{range}(C) = \text{range}(A) + \text{range}(B).
\]
(Recall that $S + T$ is the set \{z $\in \mathbb{R}^n$ | there is x $\in S$ and y $\in T$ so that z = x + y\}. In other words, z $\in S + T$ means there exists x $\in S$ and y $\in T$ satisfying z = x + y.)

c) If $S, T \subset \mathbb{R}^n$ are subspaces, then
\[
S \cap T = (S^\perp + T^\perp)^\perp.
\]
(Recall that $S^\perp = \{x \in \mathbb{R}^n \mid x^\top y = 0 \text{ for all } y \in S\}$ is the orthogonal complement of $S$.)
Using Equation (2), explain how to find a matrix $D$ satisfying
\[
\text{range}(D) = \text{range}(A) \cap \text{range}(B).
\]
(Aside: You are capable of proving Equation (2), but we omit this exercise on the exam.)

d) The file $A_{\text{and}}\_B$.json contains particular matrices $A \in \mathbb{R}^{10 \times 7}$ and $B \in \mathbb{R}^{10 \times 6}$.
Compute and report the dimension of $\text{range}(A) \cap \text{range}(B)$.

e) If $F = \begin{bmatrix} f_1 & \cdots & f_p \end{bmatrix} \in \mathbb{R}^{n \times p}$ and $G \in \mathbb{R}^{n \times m}$ are two matrices, then
\[
\text{range}(F) \subset \text{range}(G) \text{ if and only if } f_j \in \text{range}(G) \text{ for } j = 1, \ldots, p.
\]
(Aside: You are capable of proving Statement (3), but we omit this exercise on the exam.)

f) With $A$ and $B$ as in part (d), is $\text{range}(A) = \text{range}(B)$?
If so, verify it using (3) and your solution to part (e). If not, then give a vector $v \in \mathbb{R}^n$ in one space but not in the other, and verify this property of $v$ using your solution to (e).
4. True or false. State whether each of the following statements is true or false. You need not provide a proof or counterexample. These vary widely in difficulty. They are not given in order of difficulty.

a) Suppose $A \in \mathbb{R}^{n \times n}$ with $\text{rank}(A) = n$. If $A^2 = A$, then $A = I$.

b) Suppose $A \in \mathbb{R}^{n \times n}$ with $\text{rank}(A) = n$. If $A^2 = I$, then $A = I$.

c) If $Q \in \mathbb{R}^{n \times k}$ has orthonormal columns, then $\|Q^T x\| \leq \|x\|$ for all $x \in \mathbb{R}^n$.

d) Suppose $A \in \mathbb{R}^{m \times n}$ with $m > n$. If $A$ is full rank, then $(AA^T)^{-1}$ may exist.

e) Suppose $Q \succ 0$. If $Qa = b$ and $Qb = a$, then $a = b$.

f) If $A = [\begin{array}{c} a_1 \\ \vdots \\ a_n \end{array}] \in \mathbb{R}^{m \times n}$ satisfies $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^n$, then $A^T A = I$.

(We showed the converse in class. If $U \in \mathbb{R}^{m \times n}$ with $U^T U = I$, then $\|Ux\| = \|x\|$ for all $x \in \mathbb{R}^n$).

g) If $A$ is left invertible, then so is

$$\begin{bmatrix} A \\ B \end{bmatrix},$$

for any $B$.

h) Suppose $A \in \mathbb{R}^{n \times n}$ with $A^T = A$. If $A > 0$, then $A_{ij} > 0$ for all $i, j = 1, \ldots, n$.

i) Suppose $A \in \mathbb{R}^{n \times n}$ with $A^T = A$. If $A_{ij} > 0$ for all $i, j = 1, \ldots, n$, then $A > 0$.

j) If $A, B \in \mathbb{R}^{n \times n}$ are diagonal, then $e^{A+B} = e^A e^B$.

k) Suppose $Q \in \mathbb{R}^{n \times k}$ with $Q^T Q = I$. Then $\|Q\|_F = 1$.

l) Suppose $Q \in \mathbb{R}^{n \times k}$ with $Q^T Q = I$. Then $\|Q^T\| = 1$.

m) Every halfspace $H \subset \mathbb{R}^n$ is a subspace.

(Recall that a halfspace is a set $\{x \in \mathbb{R}^n \mid x^T a \leq 0\}$ for some $a \in \mathbb{R}^n$.)