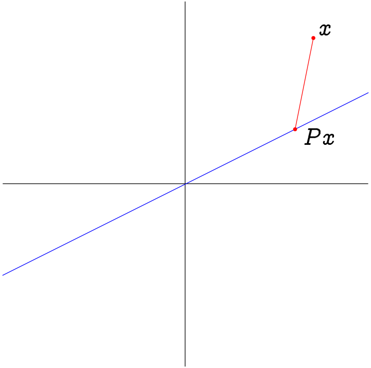
Projectors

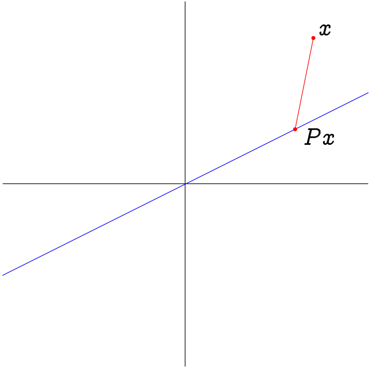
Projectors

- a square matrix P which satisfies $P^2 = P$ is called a *projector*
- we say P *projects onto* **range**(P)
- if $y \in$ **range**(P) then $Py = y$ as expected
- for any x we have $x - Px \in$ **null**(P)
  - because $P(x - Px) = Px - P^2x = 0$
- and so P *projects along* **null**(P)



Complementary projectors

- the matrix $Q = I - P$ is called a *complementary projector* to P
- it is a projector, because $Q^2 = Q$
- **null**(Q) = **range**(P) because
  - if $Qx = 0$ then $Px = x$ and so $x \in$ **range**(P)
  - if $x = Pz$ for some z then $Qx = x - Px = Pz - P^2 z = 0$
- same argument with P and Q swapped gives **null**(P) = **range**(Q)



Subspaces

- we have the intuitive idea that a projector P projects along one subspace onto another
- suggests that these subspaces have to span $\mathbb{R}^n$
- for any vector x we have
$$x = Px + (I - P)x$$
and so x is the sum of a vector in **range**(P) and a vector in **null**(P)
- these two subspaces only intersect at 0
  - because if $x = Pz$ for some z, and $Px = 0$, then $P^2z = Pz = x = 0$
- so these two subspaces are complementary

Examples

- *rank one projector:* if a is unit vector, then $P = aa^\mathsf{T}$ is a projector
- *orthogonal projector:* if U is a matrix with orthonormal columns, then $P = UU^\mathsf{T}$ is a projector
  - **range**$(P) =$ **range**(U) because if $x = Uz$ then $U^\mathsf{T}x = U^\mathsf{T}Uz = z$, and so $x = Uz = Uu^\mathsf{T}x$
- *general projector:* if $A \in \mathbb{R}^{m\times n}$ and **null**$(A) = \{0\}$, then $P = A(A^\mathsf{T}A)^{-1}A^\mathsf{T}$ is a projector
  - **range**$(P) =$ **range**(A), because $x = Az$ implies that $A^\mathsf{T}x = A^\mathsf{T}Az$ and so $z = (A^\mathsf{T}A)^{-1}A^\mathsf{T}x$, so $x = Px$

Orthogonal projectors

- if a projector P satisfies $P = P^\mathsf{T}$ it is called an *orthogonal projector*
- if $P = P^\mathsf{T}$ then **range**$(P) \perp$ **null**(P)
- because $x \in$ **range**(P) implies that $x = Pz$ for some z

  and $y \in$ **null**(P) implies $y \in$ **range**$(I - P)$ so $y = (I - P)w$ for some w

  then $x^\mathsf{T} y = z^\mathsf{T} P(I - P)w = 0$

Orthogonal projectors

- the converse is also true; if P is a projector, and **range**$(P) \perp$ **null**(P), then $P = P^{\mathsf{T}}$
- to see this, let $Q = \left[\begin{array}{cc} Q_1 & Q_2 \end{array}\right]$ be an orthogonal matrix, with **range**$(Q_1) =$ **range**(P) and **range**$(Q_2) =$ **null**(P), then

  $$Q^{\mathsf{T}} P Q = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

  and so

  $$P = Q \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Q^{\mathsf{T}}$$

  which is symmetric