Orthogonality

Stephen Boyd and Sanjay Lall

EE263
Stanford University
Orthonormal set of vectors

set of vectors \( \{u_1, \ldots, u_k\} \subset \mathbb{R}^n \) is

- **normalized** if \( \|u_i\| = 1, \ i = 1, \ldots, k \)  
  (\( u_i \) are called **unit vectors** or **direction vectors**)  

- **orthogonal** if \( u_i \perp u_j \) for \( i \neq j \)

- **orthonormal** if both

**slang:** we say ‘\( u_1, \ldots, u_k \) are orthonormal vectors’ but orthonormality (like independence) is a property of a **set** of vectors, not vectors individually

in terms of \( U = [u_1 \ \cdots \ u_k] \), orthonormal means

\[
U^T U = I_k
\]
Orthonormality

an orthonormal set of vectors is independent

- to see this, multiply $Ux = 0$ by $U^T$
- hence $\{u_1, \ldots, u_k\}$ is an orthonormal basis for

$$\text{span}(u_1, \ldots, u_k) = \text{range}(U)$$

- warning: if $k < n$ then $UU^T \neq I$ (since its rank is at most $k$)
  (more on this matrix later . . .)
**Orthonormal basis for** $\mathbb{R}^n$

A matrix $U$ is called *orthogonal* if

$U$ is square and $U^T U = I$

- the set of columns $u_1, \ldots, u_n$ is an orthonormal *basis* for $\mathbb{R}^n$
- (you’d think such matrices would be called *orthonormal*, not *orthogonal*)
- it follows that $U^{-1} = U^T$, and hence also $UU^T = I$, i.e.,

$$\sum_{i=1}^{n} u_i u_i^T = I$$
Expansion in orthonormal basis

suppose $U$ is orthogonal, so $x = UU^T x$, i.e.,

$$x = \sum_{i=1}^{n} (u_i^T x) u_i$$

- $u_i^T x$ is called the component of $x$ in the direction $u_i$
- $a = U^T x$ resolves $x$ into the vector of its $u_i$ components
- $x = Ua$ reconstitutes $x$ from its $u_i$ components
- $x = Ua = \sum_{i=1}^{n} a_i u_i$ is called the ($u_i$-) expansion of $x$
Complementary subspaces

if \( Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \) and \( Q \) is orthogonal then \( \text{range}(Q_1) \) and \( \text{range}(Q_2) \) are called \textit{complementary subspaces}, because

\[
\text{range}(Q_2) = \text{range}(Q_1)^\perp
\]

- they are orthogonal \textit{i.e.}, every vector in the first subspace is orthogonal to every vector in the second subspace
- every vector in \( \mathbb{R}^m \) can be expressed as a sum of two vectors, one from each subspace
- each subspace is the \textit{orthogonal complement} of the other
Complementary subspaces

\( \text{range}(Q_2) = \text{range}(Q_1)^\perp \)

- **range** \( Q_2 \subset (\text{range } Q_1)^\perp \) because \( Q_1^T Q_2 = 0 \)

- to show **range** \( Q_2 \supset (\text{range } Q_1)^\perp \), suppose \( x \in (\text{range } Q_1)^\perp \), then \( Q_1^T x = 0 \), and since \( x = Q_1 Q_1^T x + Q_2 Q_2^T x \) we have \( x = Q_2 Q_2^T x \) and so \( x \in \text{range } Q_2 \)
Geometric interpretation

if $U$ has orthonormal columns then transformation $w = Uz$

- preserves norm of vectors, i.e., $\|Uz\| = \|z\|
- preserves angles between vectors, i.e., $\angle(Uz, U\tilde{z}) = \angle(z, \tilde{z})$
- we say $U$ is isometric, it preserves distances