Interpreting Linear Equations

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Broad categories of applications

linear model or function $y = Ax$

some broad categories of applications:

- estimation or inversion
- control or design
- mapping or transformation

(this list is not exclusive; can have combinations . . . )
Estimation or inversion

\[ y = Ax \]

- \( y_i \) is \( i \)th measurement or sensor reading (which we know)
- \( x_j \) is \( j \)th parameter to be estimated or determined
- \( a_{ij} \) is sensitivity of \( i \)th sensor to \( j \)th parameter

Sample problems:

- Find \( x \), given \( y \)
- Find all \( x \)'s that result in \( y \) (\( i.e. \), all \( x \)'s consistent with measurements)
- If there is no \( x \) such that \( y = Ax \), find \( x \) s.t. \( y \approx Ax \) (\( i.e. \), if the sensor readings are inconsistent, find \( x \) which is almost consistent)
Control or design

\[ y = Ax \]

- \( x \) is vector of design parameters or inputs (which we can choose)
- \( y \) is vector of results, or outcomes
- \( A \) describes how input choices affect results

Sample problems:

- find \( x \) so that \( y = y_{des} \)
- find all \( x \)'s that result in \( y = y_{des} \) (i.e., find all designs that meet specifications)
- among \( x \)'s that satisfy \( y = y_{des} \), find a small one (i.e., find a small or efficient \( x \) that meets specifications)
Mapping or transformation

- $x$ is mapped or transformed to $y$ by linear function $y = Ax$

Sample problems:

- Determine if there is an $x$ that maps to a given $y$
- (if possible) find an $x$ that maps to $y$
- Find all $x$'s that map to a given $y$
- If there is only one $x$ that maps to $y$, find it (i.e., decode or undo the mapping)
Matrix multiplication as mixture of columns

write $A \in \mathbb{R}^{m \times n}$ in terms of its columns

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

where $a_j \in \mathbb{R}^m$. Then then $y = Ax$ means

$$y = x_1a_1 + x_2a_2 + \cdots + x_na_n$$

($x_j$’s are scalars, $a_j$’s are $m$-vectors)

- $y$ is a (linear) combination or mixture of the columns of $A$
- coefficients of $x$ give coefficients of mixture
- each column of $A$ represents an actuator
**Geometric interpretation of control**

example: \( A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}, \quad y = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix} \)

\[ Ax = a_1 + (-0.5)a_2 = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix} \]

another example:

\[ a_j = Ae_j \]

where \( e_j \) is the \( j \)th unit vector:

\[ e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ n \end{bmatrix} \]

\( j \)th column of \( A \) gives response to unit \( j \)th input
Matrix multiplication as inner product with rows

write \( A \) in terms of its rows:

\[
A = \begin{bmatrix}
\tilde{a}_1^T \\
\tilde{a}_2^T \\
\vdots \\
\tilde{a}_m^T 
\end{bmatrix}
\]

where \( \tilde{a}_i \in \mathbb{R}^n \)

then \( y = Ax \) can be written as

\[
y = \begin{bmatrix}
\tilde{a}_1^T x \\
\tilde{a}_2^T x \\
\vdots \\
\tilde{a}_m^T x 
\end{bmatrix}
\]

\( y_i = \tilde{a}_i^T x \), so that \( y_i \) is inner product of \( i \)th row of \( A \) with \( x \)

each row of \( A \) represents a sensor
Geometric interpretation of estimation

\[ a^T_i x = \text{constant} \]

is a (hyper-)plane in \( \mathbb{R}^n \) normal to \( a_i \).

If \( Ax = y \) then \( x \) is on intersection of hyperplanes \( a^T_i x = y_i \)

\[
A = \begin{bmatrix}
2 & 1 \\
-1 & 1 \\
\end{bmatrix}
\]

\[
x = \begin{bmatrix}
1 \\
2 \\
\end{bmatrix}
\]

\[
y = \begin{bmatrix}
4 \\
1 \\
\end{bmatrix}
\]
Matrix multiplication as composition

for $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, $C = AB \in \mathbb{R}^{m \times p}$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

composition interpretation

$y = Cz$ represents composition of $y = Ax$ and $x = Bz$

(note that $B$ is on left in block diagram)
Column and row interpretations

can write product $C = AB$ as

$$C = \begin{bmatrix} c_1 & \cdots & c_p \end{bmatrix} = AB = \begin{bmatrix} Ab_1 & \cdots & Ab_p \end{bmatrix}$$

i.e., $i$th column of $C$ is $A$ acting on $i$th column of $B$

similarly we can write

$$C = \begin{bmatrix} \tilde{c}_1^T \\ \vdots \\ \tilde{c}_m^T \end{bmatrix} = AB = \begin{bmatrix} \tilde{a}_1^T B \\ \vdots \\ \tilde{a}_m^T B \end{bmatrix}$$

i.e., $i$th row of $C$ is $i$th row of $A$ acting (on left) on $B$
Inner product interpretation

\[ c_{ij} = \bar{a}_i^T b_j = \langle \bar{a}_i, b_j \rangle \]

i.e., entries of \( C \) are inner products of rows of \( A \) and columns of \( B \)

- \( c_{ij} = 0 \) means \( i \)th row of \( A \) is orthogonal to \( j \)th column of \( B \)

- **Gram matrix** of vectors \( f_1, \ldots, f_n \) defined as \( G_{ij} = f_i^T f_j \)
  
  (gives inner product of each vector with the others)

- \( G = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}^T \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix} \)