Controllability and state transfer

- state transfer
- reachable set, controllability matrix
- minimum norm inputs
State transfer

consider $\dot{x} = Ax + Bu$ (or $x(t+1) = Ax(t) + Bu(t)$) over time interval $[t_i, t_f]$ 
we say input $u : [t_i, t_f] \rightarrow \mathbb{R}^m$ steers or transfers state from $x(t_i)$ to $x(t_f)$ (over time interval $[t_i, t_f]$) 
(subscripts stand for initial and final) 
questions:

▸ where can $x(t_i)$ be transferred to at $t = t_f$?

▸ how quickly can $x(t_i)$ be transferred to some $x_{target}$?

▸ how do we find a $u$ that transfers $x(t_i)$ to $x(t_f)$?

▸ how do we find a ‘small’ or ‘efficient’ $u$ that transfers $x(t_i)$ to $x(t_f)$?
Reachability

Consider state transfer from \( x(0) = 0 \) to \( x(t) \)
we say \( x(t) \) is \textit{reachable} (in \( t \) seconds or epochs)
we define \( \mathcal{R}_t \subseteq \mathbb{R}^n \) as the set of points reachable in \( t \) seconds or epochs
for CT system \( \dot{x} = Ax + Bu \),
\[
\mathcal{R}_t = \left\{ \int_0^t e^{(t-\tau)A}Bu(\tau) \, d\tau \mid u : [0,t] \to \mathbb{R}^m \right\}
\]
and for DT system \( x(t+1) = Ax(t) + Bu(t) \),
\[
\mathcal{R}_t = \left\{ \sum_{\tau=0}^{t-1} A^{t-1-\tau}Bu(\tau) \mid u(0), \ldots, u(t-1) \in \mathbb{R}^m \right\}
\]
Reachable set

- $\mathcal{R}_t$ is a subspace of $\mathbb{R}^n$

- $\mathcal{R}_t \subseteq \mathcal{R}_s$ if $t \leq s$
  
  (i.e., can reach more points given more time)

we define the *reachable set* $\mathcal{R}$ as the set of points reachable for some $t$:

$$\mathcal{R} = \bigcup_{t \geq 0} \mathcal{R}_t$$
Cayley-Hamilton theorem

If \( p(s) = a_0 + a_1 s + \cdots + a_k s^k \) is a polynomial and \( A \in \mathbb{R}^{n \times n} \), we define

\[
p(A) = a_0 I + a_1 A + \cdots + a_k A^k
\]

**Cayley-Hamilton theorem:** For any \( A \in \mathbb{R}^{n \times n} \) we have \( \chi(A) = 0 \), where \( \chi(s) = \det(sI - A) \)

**Example:** With \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) we have \( \chi(s) = s^2 - 5s - 2 \), so

\[
\chi(A) = A^2 - 5A - 2I
\]

\[
= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2I
\]

\[
= 0
\]
 Reachability for discrete-time LDS

DT system \( x(t + 1) = Ax(t) + Bu(t), x(t) \in \mathbb{R}^n \)

\[
\begin{bmatrix}
    u(t - 1) \\
    \vdots \\
    u(0)
\end{bmatrix}
\]

where \( C_t = [B \ AB \ \cdots \ \ A^{t-1}B] \) so reachable set at \( t \) is \( R_t = \text{range}(C_t) \)

by Cayley-Hamilton theorem, we can express each \( A^k \) for \( k \geq n \) as linear combination of \( A^0, \ldots, A^{n-1} \)

hence for \( t \geq n \), \( \text{range}(C_t) = \text{range}(C_n) \)

thus we have

\[
R_t = \begin{cases}
\text{range}(C_t) & t < n \\
\text{range}(C) & t \geq n
\end{cases}
\]

where \( C = C_n \) is called the \textit{controllability matrix}

- any state that can be reached can be reached by \( t = n \)
- the reachable set is \( R = \text{range}(C) \)
Controllable system

system is called \textit{reachable} or \textit{controllable} if all states are reachable (\(i.e., \mathcal{R} = \mathbb{R}^n\))

system is reachable if and only if \(\text{rank}(C) = n\)

\textbf{example:} \(x(t + 1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)\)

controllability matrix is \(C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\)

hence system is not controllable; reachable set is

\[\mathcal{R} = \text{range}(C) = \{ x \mid x_1 = x_2 \}\]
General state transfer

with \( t_f > t_i \),

\[
x(t_f) = A^{t_f-t_i}x(t_i) + C_{t_f-t_i} \begin{bmatrix} u(t_f - 1) \\ \vdots \\ u(t_i) \end{bmatrix}
\]

hence can transfer \( x(t_i) \) to \( x(t_f) = x_{\text{des}} \)

\[
\Leftrightarrow \quad x_{\text{des}} - A^{t_f-t_i}x(t_i) \in \mathcal{R}_{t_f-t_i}
\]

- general state transfer reduces to reachability problem
- if system is controllable any state transfer can be achieved in \( \leq n \) steps
- important special case: driving state to zero (sometimes called regulating or controlling state)
least-norm input for reachability

assume system is reachable, \( \text{rank}(C_t) = n \)

to steer \( x(0) = 0 \) to \( x(t) = x_{\text{des}} \), inputs \( u(0) , \ldots , u(t-1) \) must satisfy

\[
x_{\text{des}} = C_t \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}
\]

among all \( u \) that steer \( x(0) = 0 \) to \( x(t) = x_{\text{des}} \), the one that minimizes \( \sum_{\tau=0}^{t-1} \| u(\tau) \|^2 \) is given by

\[
\begin{bmatrix} u_{\text{ln}}(t-1) \\ \vdots \\ u_{\text{ln}}(0) \end{bmatrix} = C_t^T (C_tC_t^T)^{-1} x_{\text{des}}
\]

\( u_{\text{ln}} \) is called \textit{least-norm} or \textit{minimum energy} input that effects state transfer

can express as

\[
\begin{aligned}
\begin{bmatrix} u_{\text{ln}}(\tau) \end{bmatrix} &= B^T (A^T)^{(t-1-\tau)} \left( \sum_{s=0}^{t-1} A^s B B^T (A^T)^s \right)^{-1} x_{\text{des}}, \\
\end{aligned}
\]

for \( \tau = 0, \ldots , t - 1 \)
Minimum energy

$E_{\text{min}}$, the minimum value of $\sum_{\tau=0}^{t-1} \|u(\tau)\|^2$ required to reach $x(t) = x_{\text{des}}$, is sometimes called **minimum energy** required to reach $x(t) = x_{\text{des}}$

$$E_{\text{min}} = \sum_{\tau=0}^{t-1} \|u_{\ln}(\tau)\|^2 = (C_t^T(C_tC_t^T)^{-1}x_{\text{des}})^T C_t^T(C_tC_t^T)^{-1} x_{\text{des}}$$

$$= x_{\text{des}}(C_tC_t^T)^{-1} x_{\text{des}}$$

$$= x_{\text{des}} \left( \sum_{\tau=0}^{t-1} A^T BB^T (A^T)^{\tau} \right)^{-1} x_{\text{des}}$$

- $E_{\text{min}}(x_{\text{des}}, t)$ gives measure of how hard it is to reach $x(t) = x_{\text{des}}$ from $x(0) = 0$ (*i.e.*, how large a $u$ is required)

- $E_{\text{min}}(x_{\text{des}}, t)$ gives practical measure of controllability/reachability (as function of $x_{\text{des}}, t$)

- ellipsoid $\{ z \mid E_{\text{min}}(z, t) \leq 1 \}$ shows points in state space reachable at $t$ with one unit of energy (shows directions that can be reached with small inputs, and directions that can be reached only with large inputs)
Energy dependence on time

$\varepsilon_{\text{min}}$ as function of $t$:

if $t \geq s$ then

$$
\sum_{\tau=0}^{t-1} A^\tau BB^T(A^\tau)^\tau \geq \sum_{\tau=0}^{s-1} A^\tau BB^T(A^\tau)^\tau
$$

hence

$$
\left( \sum_{\tau=0}^{t-1} A^\tau BB^T(A^\tau)^\tau \right)^{-1} \leq \left( \sum_{\tau=0}^{s-1} A^\tau BB^T(A^\tau)^\tau \right)^{-1}
$$

so $\varepsilon_{\text{min}}(x_{\text{des}}, t) \leq \varepsilon_{\text{min}}(x_{\text{des}}, s)$

i.e.: takes less energy to get somewhere more leisurely
Example:

\[ x(t + 1) = \begin{bmatrix} 1.75 & 0.8 \\ -0.95 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \]

\[ \mathcal{E}_{\text{min}}(z, t) \text{ for } z = [1 \ 1]^T: \]
Example

ellipsoids $\mathcal{E}_{\text{min}} \leq 1$ for $t = 3$ and $t = 10$: 

\[ \mathcal{E}_{\text{min}}(x, 3) \leq 1 \]

\[ \mathcal{E}_{\text{min}}(x, 10) \leq 1 \]
Continuous-time reachability

consider now $\dot{x} = Ax + Bu$ with $x(t) \in \mathbb{R}^n$

reachable set at time $t$ is

$$\mathcal{R}_t = \left\{ \int_0^t e^{(t-\tau)A}Bu(\tau) \, d\tau \middle| u : [0, t] \to \mathbb{R}^m \right\}$$

**fact:** for $t > 0$, $\mathcal{R}_t = \mathcal{R} = \text{range}(C)$, where

$$C = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is the controllability matrix of $(A, B)$

- same $\mathcal{R}$ as discrete-time system
- for continuous-time system, any reachable point can be reached as fast as you like (with large enough $u$)
Example

- unit masses at $y_1$, $y_2$, connected by unit springs, dampers
- input is tension between masses
- state is $x = [y^T \ ȳ^T]^T$

The system is

\[
x = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} u
\]

- can we maneuver state anywhere, starting from $x(0) = 0$?
- if not, where can we maneuver state?
Example

controllability matrix is

\[ C = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 & 2 \\ 0 & -1 & 2 & -2 \\ 1 & -2 & 2 & 0 \\ -1 & 2 & -2 & 0 \end{bmatrix} \]

hence reachable set is

\[ \mathcal{R} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\} \]

we can reach states with \( y_1 = -y_2, \dot{y}_1 = -\dot{y}_2, \text{i.e., precisely the differential motions} \)

it’s obvious — internal force does not affect center of mass position or total momentum!