Conditional Gaussians

Conditional pdf

- the conditional pdf $p^{|y}$ of x given $y = y_{\text{meas}}$ is

$$p^{|y}(x, y_{\text{meas}}) = \frac{p(x, y_{\text{meas}})}{p^{y}(y_{\text{meas}})}$$

Conditional mean and covariance

- the *conditional mean* of x given y is

$$\mathbf{E}(x \mid y = y_{\mathsf{meas}}) = \int x\, p^{|y}(x, y_{\mathsf{meas}})\, dx$$

  this is a *function of y_{meas}*

- the *conditional covariance* of x given y is

$$\mathbf{cov}(x \mid y = w) = \mathbf{E}\left(\left(x - f(w)\right) \left(x - f(w)\right)^T \,\middle|\, y = w \right)$$

  here $f(w) = \mathbf{E}(x \mid y = w)$. The conditional covariance is also a function of y_{meas}

Conditional pdf for a Gaussian

Suppose $x \sim \mathcal{N}(0, \Sigma)$, and

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

Suppose we measure $x_2 = y$. We would like to find the conditional pdf of x_1 given $x_2 = y$

- Is it Gaussian?
- What is the *conditional mean* $\mathbf{E}\big(x_1 \,|\, x_2 = y\big)$ of x_1 given x_2
- What is the *conditional covariance* $\mathbf{cov}(x_1 \,|\, x_2 = y)$ of x_1 given x_2?

Conditional pdf for a Gaussian

- the pdf of x is

$$p^x(x) = c_1 \exp\left(-\frac{1}{2}x^T\Sigma^{-1}x\right)$$

- by the completion of squares formula

$$\Sigma^{-1} = \begin{bmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix} \begin{bmatrix} (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix}$$

- hence

$$x^T\Sigma^{-1}x = (x_1 - Lx_2)^T T^{-1}(x_1 - Lx_2) + x_2^T\Sigma_{22}^{-1}x_2$$

where

$$T = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \qquad L = \Sigma_{12}\Sigma_{22}^{-1}$$

Conditional pdf for a Gaussian

- the conditional pdf of x_1 given x_2 is therefore

$$p^{|x_2}(x_1, y) = \frac{c_1}{p^{x_2}(y)} \exp\left(-\frac{1}{2}(x_1 - Ly)^T T^{-1}(x_1 - Ly) - \frac{1}{2} y^T \Sigma_{22}^{-1} y \right)$$

$$= \frac{c_1}{p^{x_2}(y)} \exp\left(-\frac{1}{2} y^T \Sigma_{22}^{-1} y \right) \exp\left(-\frac{1}{2}(x_1 - Ly)^T T^{-1}(x_1 - Ly) \right)$$

- therefore $p^{|x_2}(x_1, y)$ is *Gaussian*

$$p^{|x_2}(x_1, y) = c_2(y) \exp\left(-\frac{1}{2}(x_1 - Ly)^T T^{-1}(x_1 - Ly) \right)$$

where $c_2(y)$ is such that $\displaystyle\int p^{|x_2}(x_1, y)\, dx_1 = 1$

Conditional pdf for a Gaussian

- If $x \sim \mathcal{N}(0, \Sigma)$, then the conditional pdf of x_1 given $x_2 = y$ is *Gaussian*
- The *conditional mean* is

$$\mathbf{E}(x_1 \mid x_2 = y) = \Sigma_{12}\Sigma_{22}^{-1}y$$

  it is a *linear function* of y
- the *conditional covariance* is

$$\mathbf{cov}(x_1 \mid x_2 = y) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

  it does *not* depend on y

Conditional pdf for a Gaussian

- Here

$$\mathbf{cov}(x) = \begin{bmatrix} 2 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$

- We have

$$L = 0.8 \qquad T = 1.36$$

- Hence

$$\mathbf{E}(x_1 \mid x_2 = 2) = 1.6$$

$$\mathbf{cov}(x_1 \mid x_2 = 2) = 0.8$$