Controllability and state transfer

- state transfer
- reachable set, controllability matrix
- minimum norm inputs
State transfer

consider $\dot{x} = Ax + Bu$ (or $x(t + 1) = Ax(t) + Bu(t)$) over time interval $[t_i, t_f]$ we say input $u : [t_i, t_f] \rightarrow \mathbb{R}^m$ steers or transfers state from $x(t_i)$ to $x(t_f)$ (over time interval $[t_i, t_f]$)

(subscripts stand for initial and final)

questions:

- where can $x(t_i)$ be transferred to at $t = t_f$?
- how quickly can $x(t_i)$ be transferred to some $x_{\text{target}}$?
- how do we find a $u$ that transfers $x(t_i)$ to $x(t_f)$?
- how do we find a ‘small’ or ‘efficient’ $u$ that transfers $x(t_i)$ to $x(t_f)$?
Reachability

\[
\chi(t) = e^{tA} \chi(0) + \int_0^t e^{(t-\tau)A} \mathbf{B} u(\tau) \, d\tau
\]

consider state transfer from \(x(0) = 0\) to \(x(t)\)

we say \(x(t)\) is \textit{reachable} (in \(t\) seconds or epochs)

we define \(\mathcal{R}_t \subseteq \mathbb{R}^n\) as the set of points reachable in \(t\) seconds or epochs

for CT system \(\dot{x} = Ax + Bu\),

\[
\mathcal{R}_t = \left\{ \int_0^t e^{(t-\tau)A} Bu(\tau) \, d\tau \mid u : [0, t] \to \mathbb{R}^m \right\}
\]

and for DT system \(x(t + 1) = Ax(t) + Bu(t)\),

\[
\mathcal{R}_t = \left\{ \sum_{\tau=0}^{t-1} A^{t-1-\tau} Bu(\tau) \mid u(0), \ldots, u(t-1) \in \mathbb{R}^m \right\}
\]
Reachable set

- $\mathcal{R}_t$ is a subspace of $\mathbb{R}^n$
- $\mathcal{R}_t \subseteq \mathcal{R}_s$ if $t \leq s$
  (i.e., can reach more points given more time)

we define the reachable set $\mathcal{R}$ as the set of points reachable for some $t$:

$$\mathcal{R} = \bigcup_{t \geq 0} \mathcal{R}_t$$
Cayley-Hamilton theorem

\[ \chi(s) = \det(sI - A) \rightarrow \chi(A) = \det(A - A) = \det(0) = 0 \]

if \( p(s) = a_0 + a_1s + \cdots + a_ks^k \) is a polynomial and \( A \in \mathbb{R}^{n\times n} \), we define

\[ p(A) = a_0I + a_1A + \cdots + a_kA^k \]

Cayley-Hamilton theorem: for any \( A \in \mathbb{R}^{n\times n} \) we have \( \chi(A) = 0 \), where \( \chi(s) = \det(sI - A) \)

example: with \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \) we have \( \chi(s) = s^2 - 5s - 2 \), so

\[ sI - A = \begin{bmatrix} s-1 & -2 \\ -3 & s-4 \end{bmatrix} \]

\[ \chi(A) = A^2 - 5A - 2I \]

\[ = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2I \]

\[ = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} \]

\[ = 0 \]
Reachability for discrete-time LDS

DT system $x(t + 1) = Ax(t) + Bu(t)$, $x(t) \in \mathbb{R}^n$

\[
\begin{align*}
\mathcal{R}_t &= x(t) = C_t \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}
\end{align*}
\]

where $C_t = \begin{bmatrix} B & AB & \cdots & A^{t-1}B \end{bmatrix}$ so reachable set at $t$ is $\mathcal{R}_t = \text{range}(C_t)$

by Cayley-Hamilton theorem, we can express each $A^k$ for $k \geq n$ as linear combination of $A^0, \ldots, A^{n-1}$

hence for $t \geq n$, $\text{range}(C_t) = \text{range}(C_n)$

thus we have

\[
\mathcal{R}_t = \begin{cases} 
\text{range}(C_t) & t < n \\
\text{range}(C) & t \geq n 
\end{cases}
\]

where $C = C_n$ is called the \textit{controllability matrix}

- any state that can be reached can be reached by $t = n$
- the reachable set is $\mathcal{R} = \text{range}(C)$
Controllable system

system is called \textit{reachable} or \textit{controllable} if all states are reachable (\textit{i.e.}, $\mathcal{R} = \mathbb{R}^n$)

system is reachable if and only if $\text{rank}(C) = n$

\textbf{example:} $x(t + 1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$

controllability matrix is $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

hence system is not controllable; reachable set is

$$\mathcal{R} = \text{range}(C) = \{ x \mid x_1 = x_2 \}$$
General state transfer

with $t_f > t_i$,

$$x(t_f) = A^{t_f-t_i}x(t_i) + C_{t_f-t_i} \begin{bmatrix} u(t_f - 1) \\ \vdots \\ u(t_i) \end{bmatrix}$$

hence can transfer $x(t_i)$ to $x(t_f) = x_{\text{des}}$

$$\Leftrightarrow \quad x_{\text{des}} - A^{t_f-t_i}x(t_i) \in \mathcal{R}_{t_f-t_i}$$

- general state transfer reduces to reachability problem
- if system is controllable any state transfer can be achieved in $\leq n$ steps
- important special case: driving state to zero (sometimes called regulating or controlling state)
Least-norm input for reachability

assume system is reachable, \( \text{rank}(C_t) = n \)

to steer \( x(0) = 0 \) to \( x(t) = x_{\text{des}} \), inputs \( u(0), \ldots, u(t - 1) \) must satisfy

\[
x_{\text{des}} = C_t \begin{bmatrix} u(t - 1) \\ \vdots \\ u(0) \end{bmatrix}
\]

among all \( u \) that steer \( x(0) = 0 \) to \( x(t) = x_{\text{des}} \), the one that minimizes \( \sum_{\tau=0}^{t-1} \| u(\tau) \|^2 \) is given by

\[
\begin{bmatrix} u_{\text{in}}(t - 1) \\ \vdots \\ u_{\text{in}}(0) \end{bmatrix} = C_t^T (C_t C_t^T)^{-1} x_{\text{des}}
\]

\( u_{\text{in}} \) is called least-norm or minimum energy input that effects state transfer

can express as

\[
u_{\text{in}}(\tau) = B^T (A^T)^{(t-1-\tau)} \left( \sum_{s=0}^{t-1} A^s B B^T (A^T)^s \right)^{-1} x_{\text{des}},
\]

for \( \tau = 0, \ldots, t - 1 \)
Minimum energy

$\mathcal{E}_{\min}$, the minimum value of $\sum_{\tau=0}^{t-1} ||u(\tau)||^2$ required to reach $x(t) = x_{\text{des}}$, is sometimes called **minimum energy** required to reach $x(t) = x_{\text{des}}$

$$
\mathcal{E}_{\min} = \sum_{\tau=0}^{t-1} ||u_{\text{in}}(\tau)||^2 = (C_t^T(C_tC_t^T)^{-1}x_{\text{des}})^T C_t^T(C_tC_t^T)^{-1}x_{\text{des}}
$$

$$
= x_{\text{des}}^T(C_tC_t^T)^{-1}x_{\text{des}}
$$

$$
= x_{\text{des}}^T \left( \sum_{\tau=0}^{t-1} A^\tau BB^T(A^T)^\tau \right)^{-1} x_{\text{des}}
$$

$\mathcal{E}_{\min}(x_{\text{des}}, t)$ gives measure of how hard it is to reach $x(t) = x_{\text{des}}$ from $x(0) = 0$ (i.e., how large a $u$ is required)

$\mathcal{E}_{\min}(x_{\text{des}}, t)$ gives practical measure of controllability/reachability (as function of $x_{\text{des}}, t$)

ellipsoid $\{ z \mid \mathcal{E}_{\min}(z, t) \leq 1 \}$ shows points in state space reachable at $t$ with one unit of energy (shows directions that can be reached with small inputs, and directions that can be reached only with large inputs)
Energy dependence on time

$\mathcal{E}_{\text{min}}$ as function of $t$:

if $t \geq s$ then

$$\sum_{\tau=0}^{t-1} A^\tau B B^T (A^\tau)^T \geq \sum_{\tau=0}^{s-1} A^\tau B B^T (A^\tau)^T$$

hence

$$\left(\sum_{\tau=0}^{t-1} A^\tau B B^T (A^\tau)^T\right)^{-1} \leq \left(\sum_{\tau=0}^{s-1} A^\tau B B^T (A^\tau)^T\right)^{-1}$$

so $\mathcal{E}_{\text{min}}(x_{\text{des}}, t) \leq \mathcal{E}_{\text{min}}(x_{\text{des}}, s)$

i.e.: takes less energy to get somewhere more leisurely
Example:

\[ x(t + 1) = \begin{bmatrix} 1.75 & 0.8 \\ -0.95 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \]

\( E_{\text{min}}(z, t) \) for \( z = [1 \ 1]^T \):

![Graph showing \( E_{\text{min}} \) vs \( t \)]
Example

ellipsoids $\mathcal{E}_{\text{min}} \leq 1$ for $t = 3$ and $t = 10$:

$\mathcal{E}_{\text{min}}(x, 3) \leq 1$

$\mathcal{E}_{\text{min}}(x, 10) \leq 1$
Continuous-time reachability

\[ CT : \quad e^{tA} = I + tA + \frac{t^2A^2}{2!} + \cdots \]
\[ = I_0 + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1} \]

consider now \( \dot{x} = Ax + Bu \) with \( x(t) \in \mathbb{R}^n \)

reachable set at time \( t \) is

\[ \mathcal{R}_t = \left\{ \int_0^t e^{(t-\tau)A}Bu(\tau)\,d\tau \quad \left| \quad u : [0,t] \to \mathbb{R}^m \right. \right\} \]

**fact:** for \( t > 0 \), \( \mathcal{R}_t = \mathcal{R} = \text{range}(C) \), where

\[ C = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \]

is the controllability matrix of \((A, B)\)

- same \( \mathcal{R} \) as discrete-time system
- for continuous-time system, any reachable point can be reached as fast as you like (with large enough \( u \))
Example

- unit masses at $y_1$, $y_2$, connected by unit springs, dampers
- input is tension between masses
- state is $x = [y^T \; \dot{y}^T]^T$

System is

$$
\dot{x} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
1 \\
-1
\end{bmatrix} u
$$

- can we maneuver state anywhere, starting from $x(0) = 0$?
- if not, where can we maneuver state?
Example

Controllability matrix is

\[
C = \begin{bmatrix}
B & AB & A^2B & A^3B
\end{bmatrix} = \begin{bmatrix}
0 & 1 & -2 & 2 \\
0 & -1 & 2 & -2 \\
1 & -2 & 2 & 0 \\
-1 & 2 & -2 & 0
\end{bmatrix}
\]

Hence reachable set is

\[
\mathcal{R} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}
\]

We can reach states with \( y_1 = -y_2, \dot{y}_1 = -\dot{y}_2 \), i.e., precisely the differential motions.

It's obvious — internal force does not affect center of mass position or total momentum!