Dynamic interpretation of eigenvectors

- invariant sets
- complex eigenvectors & invariant planes
- left eigenvectors
- modal form
- discrete-time stability
Dynamic interpretation

\[ \dot{x} = Ax \quad \rightarrow \quad x(t) = e^{tA} x(0) \]

suppose \( Av = \lambda v, \ v \neq 0 \)

if \( \dot{x} = Ax \) and \( x(0) = v \), then \( x(t) = e^{\lambda t} v \)

several ways to see this, e.g.,

\[
x(t) = e^{tA} v = \left( I + tA + \frac{(tA)^2}{2!} + \cdots \right) v
\]

\[
= v + \lambda t v + \frac{(\lambda t)^2}{2!} v + \cdots = \left( 1 + \lambda t + \frac{(\lambda t)^2}{2!} + \cdots \right) v
\]

\[
= e^{\lambda t} v
\]

(since \((tA)^k v = (\lambda t)^k v\))
Dynamic interpretation

- For $\lambda \in \mathbb{C}$, solution is complex (we'll interpret later); for now, assume $\lambda \in \mathbb{R}$

- If initial state is an eigenvector $\mathbf{u}$, resulting motion is very simple — always on the line spanned by $\mathbf{u}$

- Solution $\mathbf{x}(t) = e^{\lambda t} \mathbf{u}$ is called \textit{mode} of system $\dot{\mathbf{x}} = A\mathbf{x}$ (associated with eigenvalue $\lambda$)

- For $\lambda \in \mathbb{R}$, $\lambda < 0$, mode contracts or shrinks as $t \uparrow$

- For $\lambda \in \mathbb{R}$, $\lambda > 0$, mode expands or grows as $t \uparrow$
Invariant sets

A set $S \subseteq \mathbb{R}^n$ is **invariant** under $\dot{x} = Ax$ if whenever $x(t) \in S$, then $x(\tau) \in S$ for all $\tau \geq t$

i.e.: once trajectory enters $S$, it stays in $S$

**vector field interpretation:** trajectories only cut **into** $S$, never out
Invariant sets

suppose \( Au = \lambda u, \, u \neq 0, \, \lambda \in \mathbb{R} \)

- line \( \{ tv \mid t \in \mathbb{R} \} \) is invariant
  
  (in fact, ray \( \{ tv \mid t > 0 \} \) is invariant)

- if \( \lambda < 0 \), line segment \( \{ tv \mid 0 \leq t \leq a \} \) is invariant
Complex eigenvectors

Suppose $A v = \lambda v$, $v \neq 0$, $\lambda$ is complex.

For $a \in \mathbb{C}$, (complex) trajectory $a e^{\lambda t} v$ satisfies $\dot{x} = Ax$

Hence so does (real) trajectory

$$x(t) = \mathbb{R} (a e^{\lambda t} v)$$

$$v = (1 + i) v_0$$

$$\lambda \in \mathbb{R}$$

$$\nu = \nu_{re} + i \nu_{im}, \quad \lambda = \sigma + i \omega, \quad a = \alpha + i \beta$$

- Trajectory stays in invariant plane $\text{span}\{\nu_{re}, \nu_{im}\}$
- $\sigma$ gives logarithmic growth/decay factor
- $\omega$ gives angular velocity of rotation in plane
Dynamic interpretation: left eigenvectors

suppose \( w^T A = \lambda w^T \), \( w \neq 0 \)
then

\[
\frac{d}{dt}(w^T x) = w^T \dot{x} = w^T Ax = \lambda (w^T x)
\]

i.e., \( w^T x \) satisfies the DE \( d(w^T x)/dt = \lambda (w^T x) \)

hence \( w^T x(t) = e^{\lambda t} w^T x(0) \)

- even if trajectory \( x \) is complicated, \( w^T x \) is simple
- if, e.g., \( \lambda \in \mathbb{R} \), \( \lambda < 0 \), halfspace \( \{ z \mid w^T z \leq a \} \) is invariant (for \( a \geq 0 \))
- for \( \lambda = \sigma + i\omega \in \mathbb{C} \), \((\Re\omega)^T x \) and \((\Im\omega)^T x \) both have form

\[
e^{\sigma t} (\alpha \cos(\omega t) + \beta \sin(\omega t))
\]
**Summary**

- **right eigenvectors** are initial conditions from which resulting motion is simple (i.e., remains on line or in plane)

- **left eigenvectors** give linear functions of state that are simple, for any initial condition
Example

\[ \dot{x} = \begin{bmatrix} -1 & -10 & -10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x \]

block diagram:

![Block Diagram](image)

eigenvalues are \(-1, \pm i\sqrt{10}\)
**Example**

trajectory with $x(0) = (0, -1, 1)$:
Example

left eigenvector associated with eigenvalue $-1$ is

\[ g = \begin{bmatrix} 0.1 \\ 0 \\ 1 \end{bmatrix} \]

let’s check $g^T x(t)$ when $x(0) = (0, -1, 1)$ (as above):

\[ g^T x(t) = 0.1 x_1(t) + x_3(t) \]
Example

eigenvector associated with eigenvalue $i\sqrt{10}$ is

$$v = \begin{bmatrix} -0.554 + 0.771i \\ 0.244 + 0.175i \\ 0.055 - 0.077i \end{bmatrix}$$

so an invariant plane is spanned by

$$v_{\text{re}} = \begin{bmatrix} -0.554 \\ 0.244 \\ 0.055 \end{bmatrix}, \quad v_{\text{im}} = \begin{bmatrix} 0.771 \\ 0.175 \\ -0.077 \end{bmatrix}$$
Example

For example, with $x(0) = v_{re}$ we have
Example: Markov chain

probability distribution satisfies \( p(t + 1) = Pp(t) \)

\[ p_i(t) = \text{Prob}( z(t) = i ) \text{ so } \sum_{i=1}^{n} p_i(t) = 1 \]

\[ P_{ij} = \text{Prob}( z(t + 1) = i \mid z(t) = j ) , \text{ so } \sum_{i=1}^{n} P_{ij} = 1 \]

(such matrices are called \textit{stochastic})

rewrite as:

\[ [1 \ 1 \ \cdots \ 1]P = [1 \ 1 \ \cdots \ 1] \]

i.e., \([1 \ 1 \ \cdots \ 1]\) is a left eigenvector of \( P \) with e.v. 1

hence \( \text{det}(I - P) = 0 \), so there is a right eigenvector \( \nu \neq 0 \) with \( P\nu = \nu \)

it can be shown that \( \nu \) can be chosen so that \( \nu_i \geq 0 \), hence we can normalize \( \nu \) so that \( \sum_{i=1}^{n} \nu_i = 1 \)

\textbf{interpretation:} \( \nu \) is an \textit{equilibrium distribution}; i.e., if \( p(0) = \nu \) then \( p(t) = \nu \) for all \( t \geq 0 \)

(if \( \nu \) is unique it is called the \textit{steady-state distribution} of the Markov chain)
Modal form

suppose $A$ is diagonalizable by $T$
define new coordinates by $x = T \xi$, so

\[ T \dot{\xi} = AT \xi \quad \Leftrightarrow \quad \dot{\xi} = T^{-1} AT \xi \quad \Leftrightarrow \quad \dot{\xi} = \Lambda \xi \]

\[
\begin{bmatrix}
\dot{x}_1 \\
\vdots \\
\dot{x}_n 
\end{bmatrix} = \begin{bmatrix}
\lambda_1 & & \\
& \ddots & \\
& & \lambda_n
\end{bmatrix} \begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix} \Rightarrow \begin{cases}
\dot{x}_1 = \lambda_1 x_1 \\
\vdots \\
\dot{x}_n = \lambda_n x_n
\end{cases}
\]
Modal form

in new coordinate system, system is diagonal (decoupled):

\[ \dot{x}_1 \]

\[ \lambda_1 \]

\[ \vdots \]

\[ \lambda_n \]

\[ \dot{x}_n \]

trajectories consist of \( n \) independent modes, \( i.e., \)

\[ \dot{x}_i(t) = e^{\lambda_i t} x_i(0) \]

hence the name \textit{modal form}
Real modal form

when eigenvalues (hence $T'$) are complex, system can be put in real modal form:

$$S^{-1}AS = \text{diag}(\Lambda_r, M_{r+1}, M_{r+3}, \ldots, M_{n-1})$$

where $\Lambda_r = \text{diag}(\lambda_1, \ldots, \lambda_r)$ are the real eigenvalues, and

$$M_j = \begin{bmatrix} \sigma_j & \omega_j \\ -\omega_j & \sigma_j \end{bmatrix}, \quad \lambda_j = \sigma_j + i\omega_j, \quad j = r+1, r+3, \ldots, n$$

where $\lambda_j$ are the complex eigenvalues (one from each conjugate pair)
Real modal form

block diagram of ‘complex mode’:
Diagonalization simplifies many matrix expressions

powers (i.e., discrete-time solution):

\[ A^k = (T\Lambda T^{-1})^k \]
\[ = (T\Lambda T^{-1}) \cdots (T\Lambda T^{-1}) \]
\[ = T\Lambda^k T^{-1} \]
\[ = T \text{diag}(\lambda_1^k, \ldots, \lambda_n^k) T^{-1} \]

(for \( k < 0 \) only if \( A \) invertible, i.e., all \( \lambda_i \neq 0 \))
Diagonalization

\[ e^A = I + A + A^2 / 2! + \cdots \]
\[ = I + T \Lambda T^{-1} + (T \Lambda T^{-1})^2 / 2! + \cdots \]
\[ = T(\Lambda + \Lambda^2 / 2! + \cdots)T^{-1} \]
\[ = Te^{\Lambda}T^{-1} \]
\[ = T \text{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})T^{-1} \]
Analytic function of a matrix

for any analytic function \( f : \mathbb{R} \rightarrow \mathbb{R} \), \( i.e., \) given by power series

\[
f(a) = \beta_0 + \beta_1 a + \beta_2 a^2 + \beta_3 a^3 + \cdots
\]

we can define \( f(A) \) for \( A \in \mathbb{R}^{n \times n} \) \( i.e., \) overload \( f \) as

\[
f(A) = \beta_0 I + \beta_1 A + \beta_2 A^2 + \beta_3 A^3 + \cdots
\]

substituting \( A = T \Lambda T^{-1} \), we have

\[
f(A) = \beta_0 I + \beta_1 A + \beta_2 A^2 + \beta_3 A^3 + \cdots
\]

\[
= \beta_0 T T^{-1} + \beta_1 T \Lambda T^{-1} + \beta_2 (T \Lambda T^{-1})^2 + \cdots
\]

\[
= T \left( \beta_0 I + \beta_1 \Lambda + \beta_2 \Lambda^2 + \cdots \right) T^{-1}
\]

\[
= T \ diag(f(\lambda_1), \ldots, f(\lambda_n)) T^{-1}
\]
Solution via diagonalization

Assume $A$ is diagonalizable.

Consider LDS $\dot{x} = Ax$, with $T^{-1}AT = \Lambda$

Then

$$e^{tA} = Te^{\Lambda t}T^{-1}$$

$$T = \begin{bmatrix} \nu_1 & \ldots & \nu_n \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} \nu_1^T & \ldots & \nu_n^T \end{bmatrix}$$

$$x(t) = e^{tA}x(0) = Te^{\Lambda t}T^{-1}x(0)$$

$$= \sum_{i=1}^{n} e^{\lambda_i t} (w_i^T x(0)) \nu_i$$

Thus: any trajectory can be expressed as linear combination of modes
Interpretation

- (left eigenvectors) decompose initial state $x(0)$ into modal components $w_i^T x(0)$
- $e^{\lambda_i t}$ term propagates $i$th mode forward $t$ seconds
- reconstruct state as linear combination of (right) eigenvectors
Qualitative behavior of \( x(t) \)

\[ x = \alpha_1 u_1 + \ldots + \alpha_n u_n \]

\[ = \begin{bmatrix} u_1 & \ldots & u_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \]

\[ = T \alpha \quad \rightarrow \quad \alpha = T^{-1} x = \begin{bmatrix} \omega_1^T \\ \vdots \\ \omega_n^T \end{bmatrix} \lambda \]

\( T = \begin{bmatrix} u_1 & \ldots & u_n \end{bmatrix} \quad T^{-1} = \begin{bmatrix} \omega_1^T \\ \omega_n^T \end{bmatrix} \)

- eigenvalues give exponents that can occur in exponentials
- real eigenvalue \( \lambda \) corresponds to an exponentially decaying or growing term \( e^{\lambda t} \) in solution
- complex eigenvalue \( \lambda = \sigma + i\omega \) corresponds to decaying or growing sinusoidal term \( e^{\sigma t} \cos(\omega t + \phi) \) in solution
Qualitative behavior of $x(t)$

- $\Re \lambda_j$ gives exponential growth rate (if $> 0$), or exponential decay rate (if $< 0$) of term
- $\Im \lambda_j$ gives frequency of oscillatory term (if $\neq 0$)

![Diagram showing eigenvalues in the complex plane, with axes $\Re s$ and $\Im s$.]
Application

for what \( x(0) \) do we have \( x(t) \to 0 \) as \( t \to \infty \)?

divide eigenvalues into those with negative real parts

\[
\Re \lambda_1 < 0, \ldots, \Re \lambda_s < 0,
\]

and the others,

\[
\Re \lambda_{s+1} \geq 0, \ldots, \Re \lambda_n \geq 0
\]

from

\[
x(t) = \sum_{i=1}^{n} e^{\lambda_i t} (w_i^T x(0)) v_i
\]

condition for \( x(t) \to 0 \) is:

\[
x(0) \in \text{span}\{v_1, \ldots, v_s\}, \quad \rightarrow \text{stable subspace}
\]

or equivalently,

\[
w_i^T x(0) = 0, \quad i = s + 1, \ldots, n
\]

(can you prove this?)

(stable system: \( \text{stable subspace} = \mathbb{R}^n \))
**Stability of discrete-time systems**

suppose $A$ diagonalizable

consider discrete-time LDS $x(t + 1) = Ax(t)$

if $A = T \Lambda T^{-1}$, then $A^k = T \Lambda^k T^{-1}$

then

$$e^t A x(0)$$

$$x(t) = A^t x(0) = \sum_{i=1}^{n} \lambda_i^t (w_i^\top x(0))v_i \to 0 \quad \text{as } t \to \infty$$

for all $x(0)$ if and only if

$$|\lambda_i| < 1, \quad i = 1, \ldots, n.$$ 

we will see later that this is true even when $A$ is not diagonalizable, so we have

**fact:** $x(t + 1) = Ax(t)$ is stable if and only if all eigenvalues of $A$ have magnitude less than one