Autonomous linear dynamical systems

- autonomous linear dynamical systems
- examples
- higher order systems
- linearization near equilibrium point
- linearization along trajectory
LDS: \[
\begin{align*}
\dot{x} &= Ax + Bu \\
\text{state} &\uparrow \text{input} \\
y &= Cx + Du
\end{align*}
\]
\[
\begin{align*}
x(t+1) &= Ax(t) + Bu(t) \\
y(t) &= Cx + Du
\end{align*}
\]

Systems with inputs:

\[
\begin{array}{c|c|c}
\text{Autonomous:} \quad (u=0) & \text{DT} & \text{CT} \\
\hline
x(t) &= A^t x(0) & ? \\
\end{array}
\]
Autonomous linear dynamical systems

goes by itself: no input

continuous-time autonomous LDS has form

\[ \dot{x} = Ax + Bu \]

- \( x(t) \in \mathbb{R}^n \) is called the state
- \( n \) is the state dimension or (informally) the number of states
- \( A \) is the dynamics matrix
  (system is time-invariant if \( A \) doesn’t depend on \( t \))

\[ A = a \in \mathbb{R} \rightarrow \dot{x} = ax \]

\[ x(t) = x(0)e^{at} \]

qualitative behaviour:

\( a = 0 \): constant

\( a > 0 \): growing exponential

\( a < 0 \): shrinking exponential
Phase plane

\[ \dot{x}(t) = Ax(t) \]

\( \mathbf{x}(10) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \)

\( \dot{x}(10) = A\mathbf{x}(10) \)
Example 1

\[ \dot{x} = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} x \]

- Where you are
- Where you're going
- How fast

Vectors show:
1. Start
2. Vector field
3. Slow down
4. Accelerate
5. "where you are"
6. "where you're going"
7. "how fast"
Example 2

\[
\dot{x} = \begin{bmatrix} -0.5 & 1 \\ -1 & 0.5 \end{bmatrix} x
\]

elliptic trajectories
Block diagram representation of $\dot{x} = Ax$:

- $\int$ block represents $n$ parallel scalar integrators
- Coupling comes from dynamics matrix $A$
Block diagram

useful when $A$ has structure, e.g., block upper triangular:

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} x$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$$

$$\begin{cases} \dot{x}_1 = A_{11} x_1 + A_{12} x_2 \\ \dot{x}_2 = \frac{A_{12} x_2}{A_{22}} \end{cases}$$

here $x_1$ doesn’t affect $x_2$ at all
Linear circuit

circuit equations are

\[
\begin{align*}
C \frac{dv_c}{dt} &= i_c, \\
L \frac{di_l}{dt} &= v_l,
\end{align*}
\]

with state \( x = \begin{bmatrix} v_c \\ i_l \end{bmatrix} \), we have
\[
\dot{x} = \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} F x
\]
Chemical reactions

- reaction involving \( n \) chemicals; \( x_i \) is concentration of chemical \( i \)

- linear model of reaction kinetics
  \[
  \frac{dx_i}{dt} = a_{i1}x_1 + \cdots + a_{in}x_n
  \]

- good model for some reactions; \( A \) is usually sparse
Example

series reaction $A \xrightarrow{k_1} B \xrightarrow{k_2} C$ with linear dynamics

$$1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

plot for $k_1 = k_2 = 1$, initial $x(0) = (1, 0, 0)$

columns sum is zero

total mass at time $t = x_1(t) + x_2(t) + x_3(t)$

$$= \mathbb{1}^T x = m(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\frac{d}{dt} m(t) = \frac{d}{dt} \mathbb{1}^T x = \mathbb{1}^T \dot{x} = \mathbb{1}^T (Ax) = (\mathbb{1}^T A)x = \mathbb{0}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -k_1 & 0 & 0 \\ 0 & -k_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_1(t) = x_1(0) e^{-k_1 t}$$

$$x_2(t) = e^{-k_2 t}$$

sum of the second row of $A$
Finite-state discrete-time Markov chain

\[ x(t+1) = A x(t) = A^2 x(t-1) = A^3 x(t-2) = \ldots = A^{t+1} x(0) \]

\( z(t) \in \{1, \ldots, n\} \) is a random sequence with

\[ \text{Prob}( z(t+1) = i \mid z(t) = j ) = P_{ij} \]

where \( P \in \mathbb{R}^{n \times n} \) is the matrix of transition probabilities

can represent probability distribution of \( z(t) \) as \( n \)-vector

\[ p(t) = \begin{bmatrix} \text{Prob}(z(t) = 1) \\ \vdots \\ \text{Prob}(z(t) = n) \end{bmatrix} \]

(so, e.g., \( \text{Prob}(z(t) = 1, 2, \text{ or } 3) = [1 \ 1 \ 1 \ 0 \ldots 0] p(t) \))

then we have \( p(t+1) = P p(t) \)
Graphical representation

\[ p(t+1) = A \pi p(t) \]

\[ \pi = A \pi \rightarrow \pi \text{ eigenvector associated with } \lambda = 1 \]

\[ P(0) \approx \begin{bmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix} \]

\[ P(100) = P^{100} \]

\[ I = A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

\[ p(t) = A^t p(0) = \begin{cases} t \text{ even } & P(0) \\ t \text{ odd } & A P(0) \end{cases} \]

\( P \) is often sparse; Markov chain is depicted graphically

- nodes are states
- edges show transition probabilities
Example: Markov chain

- state 1 is ‘system OK’
- state 2 is ‘system down’
- state 3 is ‘system being repaired’

\[
p(t + 1) = \begin{bmatrix} 0.9 & 0.7 & 1.0 \\ 0.1 & 0.1 & 0 \\ 0 & 0.2 & 0 \end{bmatrix} p(t)
\]
Numerical integration of continuous system

\[
\frac{h}{\chi(0) \rightarrow \chi(h) \rightarrow \chi(2h) \rightarrow \ldots \rightarrow \chi(kh)}
\]

compute approximate solution of \( \dot{x} = Ax \), \( x(0) = x_0 \)

suppose \( h \) is small time step (\( x \) doesn’t change much in \( h \) seconds)

simple (‘forward Euler’) approximation:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) \\
\frac{dx}{dt} &= \alpha x \\
x(t+h) &\approx x(t) + h\dot{x}(t) = (I + hA)x(t) \\
x(kh) &\approx (I + hA)^k x(0)
\end{align*}
\]

by carrying out this recursion (discrete-time LDS), starting at \( x(0) = x_0 \), we get approximation

\[
x(kh) \approx (I + hA)^k x(0)
\]

(forward Euler is never used in practice)
Higher order linear dynamical systems

\[ x^{(k)} = A_{k-1}x^{(k-1)} + \cdots + A_1x^{(1)} + A_0x, \quad x(t) \in \mathbb{R}^n \]

where \( x^{(m)} \) denotes \( m \)th derivative

define new variable \( z = \begin{bmatrix} x \\ x^{(1)} \\ \vdots \\ x^{(k-1)} \end{bmatrix} \in \mathbb{R}^{nk} \), so

\[
\begin{bmatrix}
\dot{x} \\
\dot{x}^{(1)} \\
\vdots \\
\dot{x}^{(k-1)} \\
\dot{x}^{(k)}
\end{bmatrix} = \begin{bmatrix}
\ddot{x} \\
\ddot{x}^{(1)} \\
\vdots \\
\ddot{x}^{(k-1)} \\
\ddot{x}^{(k)}
\end{bmatrix} = \begin{bmatrix}
0 & I & 0 & \cdots & 0 \\
0 & 0 & I & \cdots & 0 \\
\vdots & & & & \vdots \\
0 & 0 & 0 & \cdots & I \\
A_0 & A_1 & A_2 & \cdots & A_{k-1}
\end{bmatrix}
\begin{bmatrix}
x^{(1)} \\
x^{(2)} \\
\vdots \\
x^{(k-1)} \\
x^{(k)}
\end{bmatrix}
\]

\( \dot{z} = Az \)

A (first order) LDS (with bigger state)
Higher order linear dynamical systems

\[ x^{(k)} = A_{k-1}x^{(k-1)} + \cdots + A_1x^{(1)} + A_0x, \quad x(t) \in \mathbb{R}^n \]
Mechanical systems

mechanical system with \( k \) degrees of freedom undergoing small motions:

\[ M\ddot{q} + D\dot{q} + Kq = 0 \]

- \( q(t) \in \mathbb{R}^k \) is the vector of generalized displacements
- \( M \) is the \textit{mass matrix}
- \( K \) is the \textit{stiffness matrix}
- \( D \) is the \textit{damping matrix}

with state \( x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \) we have

\[
\begin{align*}
\ddot{q} &= -D\dot{q} - Kq \\
\dot{q} &= -M^{-1}D\dot{q} - M^{-1}Kq \\
\dot{x} &= \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x
\end{align*}
\]
Linearization near equilibrium point

nonlinear, time-invariant differential equation (DE):

\[ \dot{x} = f(x) \]

where \( f : \mathbb{R}^n \to \mathbb{R}^n \)

suppose \( x_e \) is an equilibrium point, i.e., \( f(x_e) = 0 \)

(\( x(t) = x_e \) satisfies DE)

now suppose \( x(t) \) is near \( x_e \), so

\[ \dot{x}(t) = f(x(t)) \approx f(x_e) + Df(x_e)(x(t) - x_e) \]

with \( \delta x(t) = x(t) - x_e \), rewrite as

\[ \delta x(t) \approx Df(x_e)\delta x(t) \]

replacing \( \approx \) with \( = \) yields linearized approximation of DE near \( x_e \)

we hope solution of \( \delta x = Df(x_e)\delta x \) is a good approximation of \( x - x_e \)

(more later)
Example: Pendulum

\[ \ddot{\theta} = \frac{-g}{l} \sin \theta = f(\theta) \]

2nd order nonlinear DE \[ ml^2 \ddot{\theta} = -mg \sin \theta \]
rewrite as first order DE with state \( x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \):

\[ \dot{x} = \begin{bmatrix} x_2 \\ -(g/l) \sin x_1 \end{bmatrix} = f(x) \]

equilibrium point (pendulum down): \( x = 0 \)
linearized system near \( x_e = 0 \): \[ \delta x = \begin{bmatrix} 0 & 1 \\ +g/l & 0 \end{bmatrix} \delta x \]

\[ f(x) = 0 \Rightarrow \begin{bmatrix} x_2 \\ -(g/l) \sin x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_2 = 0 \\ \sin x_1 = 0 \Rightarrow x_1 = 0, \pi, 2\pi, \ldots \end{cases} \]
Does linearization ‘work’?

the linearized system usually, but not always, gives a good idea of the system behavior near $x_e$

**example 1:** $\dot{x} = -x^3$ near $x_e = 0$

for $x(0) > 0$ solutions have form $x(t) = (x(0)^{-2} + 2t)^{-1/2}$

linearized system is $\dot{x} = 0$; solutions are constant

**example 2:** $\dot{z} = z^3$ near $z_e = 0$

for $z(0) > 0$ solutions have form $z(t) = (z(0)^{-2} - 2t)^{-1/2}$

(finite escape time at $t = z(0)^{-2}/2$)

linearized system is $\dot{z} = 0$; solutions are constant
Does linearization ‘work’?

- systems with very different behavior have same linearized system
- linearized systems do not predict qualitative behavior of either system
Linearization along trajectory

1. Suppose $x_{\text{traj}} : \mathbb{R}_+ \to \mathbb{R}^n$ satisfies $\dot{x}_{\text{traj}}(t) = f(x_{\text{traj}}(t), t)$

2. Suppose $x(t)$ is another trajectory, i.e., $\dot{x}(t) = f(x(t), t)$, and is near $x_{\text{traj}}(t)$

3. Then

\[
\frac{d}{dt}(x - x_{\text{traj}}) = f(x, t) - f(x_{\text{traj}}, t) \approx D_x f(x_{\text{traj}}, t)(x - x_{\text{traj}})
\]

4. (time-varying) LDS

\[
\dot{x} = D_x f(x_{\text{traj}}, t)\delta x
\]

is called linearized or variational system along trajectory $x_{\text{traj}}$
Example: Linearized oscillator

suppose $x_{\text{traj}}(t)$ is $T$-periodic solution of nonlinear DE:

$$\dot{x}_{\text{traj}}(t) = f(x_{\text{traj}}(t)), \quad x_{\text{traj}}(t + T) = x_{\text{traj}}(t)$$

linearized system is

$$\delta\dot{x} = A(t)\delta x$$

where $A(t) = Df(x_{\text{traj}}(t))$

$A(t)$ is $T$-periodic, so linearized system is called $T$-periodic linear system.

used to study:

- startup dynamics of clock and oscillator circuits
- effects of power supply and other disturbances on clock behavior