Example: Least-squares navigation

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Navigation from range measurements

Navigation using range measurements from distant beacons

beacons far from unknown position $x \in \mathbb{R}^2$, so linearization around $x = 0$ (say) nearly exact
Navigation

ranges $y \in \mathbb{R}^4$ measured, with measurement noise $\nu$:

$$y = \begin{bmatrix} k_1^T \\ k_2^T \\ k_3^T \\ k_4^T \end{bmatrix} x + \nu$$

where $k_i$ is unit vector from 0 to beacon $i$

► problem: estimate $x \in \mathbb{R}^2$, given $y \in \mathbb{R}^4$

► measurement errors are independent, Gaussian, with standard deviation 2 (details not important, roughly a 2:1 measurement redundancy ratio)

► actual position is $x = (5.59, 10.58)$;

► measurement is $y = (-11.95, -2.84, -9.81, 2.81)$
Just enough measurements method

\[
\begin{align*}
\| \mathbf{y} - \mathbf{A} \hat{x} \| \\
\hat{x} = \mathbf{B}_{je} \mathbf{y}
\end{align*}
\]

\( y_1 \) and \( y_2 \) suffice to find \( x \) (when \( v = 0 \))

compute estimate \( \hat{x} \) by inverting top \((2 \times 2)\) half of \( A \):

\[
\begin{align*}
\| \mathbf{x} - \hat{x} \| \\
\hat{x} = \mathbf{B}_{je} \mathbf{y}
\end{align*}
\]

(norm of error: \( 3.07 \))

\[
\mathbf{B}_{je} = \begin{bmatrix} \mathbf{C}^{-1} & \mathbf{0} \end{bmatrix}
\]

\[
\mathbf{B}_{je} \mathbf{A} = \begin{bmatrix} \mathbf{C}^{-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{C} \end{bmatrix} = \mathbf{C}^{-1} \mathbf{C} + \mathbf{0} \cdot \mathbf{D} = \mathbf{I} \rightarrow \mathbf{B}_{je} \text{ a left inverse of } \mathbf{A}
\]
Least-squares method

\[ A^\dagger = (A^T A)^{-1} A^T y \]

\[ x - \hat{x} = \begin{bmatrix} B \varepsilon \end{bmatrix} \]

\[ \varepsilon = \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix} \]

compute estimate \( \hat{x} \) by least-squares:

\[ \hat{x} = A^\dagger y = \begin{bmatrix} -0.23 & -0.48 & 0.04 & 0.44 \\ -0.47 & -0.02 & -0.51 & -0.18 \end{bmatrix} y = \begin{bmatrix} 4.95 \\ 10.26 \end{bmatrix} \]

(norm of error: 0.72)

- \( B_{je} \) and \( A^\dagger \) are both left inverses of \( A \)
- larger entries in \( B \) lead to larger estimation error

for any \( B \) with \( BA = I \), we have

\[ \sum_{i,j} B_{ij}^2 \geq \sum_{i,j} A_{ij}^\dagger^2 \]