Range and Null Space

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Nullspace of a matrix

$\underline{\underline{S}}: \begin{cases} x_1, x_2 \in S: & x_1 + x_2 \in S \\ x_1 \in S, \alpha \in \mathbb{R}: & \alpha x_1 \in S \end{cases}$

$\begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix} = 0$

the *nullspace* of $A \in \mathbb{R}^{m \times n}$ is defined as

$$\textbf{null}(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

$A(x_1 + x_2) = \underbrace{Ax_1}_{0} + \underbrace{Ax_2}_{0} = 0$

$A(\alpha x_1) = \alpha(Ax_1) = 0$

- $\textbf{null}(A)$ is set of vectors mapped to zero by $y = Ax$
- $\textbf{null}(A)$ is set of vectors orthogonal to all rows of A

$\underline{\underline{\{0\}}}$

$\textbf{null}(A)$ gives *ambiguity* in x given $y = Ax$:

$y = A\underline{\underline{x}} = A(x+z)$

$= Ax + \underbrace{Az}_{0}$

- if $y = Ax$ and $z \in \textbf{null}(A)$, then $y = A(x + z)$
- conversely, if $\underline{\underline{y = Ax}}$ and $\underline{\underline{y = A\tilde{x}}}$, then $\boxed{\tilde{x} = x + z}$ for some $z \in \textbf{null}(A)$

$z = \tilde{x} - x$

$\textbf{null}(A)$ is also written $\mathcal{N}(A)$

$Az = A(\tilde{x} - x) = A\tilde{x} - Ax = y - y = 0$

$y_{des} = A\underline{\underline{x}} = A(x+z)$

$z \in \text{null}(A)$

Zero nullspace

$A \in \mathbb{R}^{10 \times 20}$

$\begin{bmatrix} a_1 & a_2 & \cdots & a_{20} \end{bmatrix}$ $a_i \in \mathbb{R}^{10}$

A is called *one-to-one* if 0 is the only element of its nullspace

skinny / ~~fat~~

$$\boxed{\textbf{null}(A) = \{0\}}$$

$x_1 \to y$, $x_2 \to y$

$y = Ax_1 = Ax_2$

$\Rightarrow A\underbrace{(x_1 - x_2)}_{z} = 0$

Equivalently,

- x can always be uniquely determined from $y = Ax$
  ($i.e.$, the linear transformation $y = Ax$ doesn't 'lose' information)
- mapping from x to Ax is one-to-one: different x's map to different y's
- columns of A are independent (hence, a basis for their span)
- ✓ A has a *left inverse*, $i.e.$, there is a matrix $B \in \mathbb{R}^{n\times m}$ s.t. $BA = I$
- ✓ $A^{\mathsf{T}}A$ is invertible

lin. indep.

$Ax = 0$ $\to$ $x = 0$ $\leftarrow$ null$(A) = \{0\}$

$\underset{n\times m}{A^{T}}\ \underset{m\times n}{A} \in \mathbb{R}^{n\times n}$

Zero nullspace

$A \in \mathbb{R}^{10\times 20}$

$A = B$

$\boxed{\begin{array}{l} A \subset B \\ B \subset A \end{array}}$

$BA = I$

$x \neq 0$

$Ax = 0$

$\underbrace{BA}_{I}x = B\cdot 0 = 0 \to x = 0$

$(A^TA)^{-1}$

- if A has a left inverse then **null**$(A) = \{0\}$ (proof by contradiction)
- **null**$(A) =$ **null**$(A^\top A)$
- if **null**$(A) = \{0\}$ then A is left invertible, because $A^\top A$ is invertible, so $\boxed{B = (A^\top A)^{-1}A^\top}$ is a left inverse

$BA = \underbrace{(A^TA)^{-1}}\underbrace{A^TA} = I$

$*\ x \in \text{null}(A) \to Ax = 0 \to (A^TA)x = 0 \to x \in \text{null}(A^TA) \to \text{null}(A) \subset \text{null}(A^TA)$

$*\ x \in \text{null}(A^TA) \to A^TAx = 0 \to x^TA^TAx = 0 \Rightarrow (Ax)^T(Ax) = 0 \Rightarrow \|Ax\|^2 = 0$

$u^Tu = \|u\|^2$

$Ax = 0$

$\Rightarrow x \in \text{null}(A)$

$\Rightarrow \text{null}(A^TA) \subset \text{null}(A)$

$BAx = 0 \not\Rightarrow Ax = 0$

Two interpretations of nullspace

suppose $z \in \textbf{null}(A)$, and $y = Ax$ represents *measurement* of x

- z is undetectable from sensors — get zero sensor readings
- x and $x + z$ are indistinguishable from sensors: $Ax = A(x + z)$

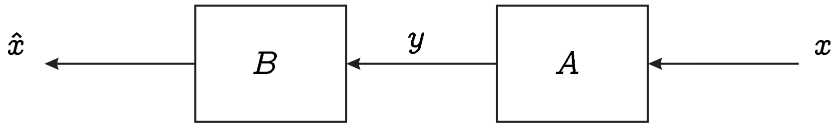
$\textbf{null}(A)$ characterizes *ambiguity* in x from measurement $y = Ax$

alternatively, if $y = Ax$ represents *output* resulting from input x

- z is an input with no result
- x and $x + z$ have same result

$\textbf{null}(A)$ characterizes *freedom of input choice* for given result

Left invertibility and estimation



$B = (A^T A)^{-1} A^T$

- apply left-inverse B at output of A
- then estimate $\hat{x} = BAx = x$ as desired
- *non-unique:* both B and C are left inverses of A

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

$BA = CA = I$

Range of a matrix

$y_1, y_2 \in \text{range}(A)$

$\Downarrow$

$Ax_1 = y_1 \ , \ Ax_2 = y_2$

$A(x_1 + x_2) = y_1 + y_2$

$y_1 + y_2 \in \text{range}(A)$

the *range* of $A \in \mathbb{R}^{m \times n}$ is defined as

$$\begin{bmatrix} \cancel{y_1} \\ y_2 \\ \vdots \\ y_{20} \end{bmatrix} \in \mathbb{R}^{19} \qquad \begin{bmatrix} \cancel{a_1^T} \\ a_2^T \\ \vdots \\ a_{20}^T \end{bmatrix}$$

$$\textbf{range}(A) = \{Ax \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

range(A) can be interpreted as

- the set of vectors that can be 'hit' by linear mapping $y = Ax$
- the span of columns of A $\quad y \in \mathbb{R}^{20} \notin \text{range}(A)$
- ✓ the set of vectors y for which $\underline{Ax = y}$ has a solution

possible outputs I <u>can</u> generate

possible measurement

range(A) is also written $\mathcal{R}(A)$

$x \in \mathbb{R}^{10}$, $y \in \mathbb{R}^{20}$ $\rightarrow$ 20 measurements / sensors

10 inputs

exactly 1 sensor is failing! $\rightarrow$ HOW TO FIND IT?

Onto matrices

$\begin{cases} \text{null}(A) = \{x \mid Ax = 0\} \subseteq \mathbb{R}^n \\ \text{range}(A) = \{Ax \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m \end{cases}$ $\quad A \in \mathbb{R}^{m\times n}$

A is called *onto* if **range**$(A) = \mathbb{R}^m$ $\rightarrow$ $\text{rank}(A) = m$

equivalently,

- $Ax = y$ can be solved in x for any y
- columns of A span $\mathbb{R}^m$
- ✓ A has a *right inverse*, *i.e.*, there is a matrix $B \in \mathbb{R}^{n\times m}$ s.t. $AB = I$ $\rightarrow$ $B^T A^T = I$
- rows of A are independent $\rightarrow$ $A^{m\times n}$ $\rightarrow$ $n \geq m$
- **null**$(A^\top) = \{0\}$
- $AA^\top$ is invertible

$\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} \in \mathbb{R}^{4\times 3}$

$\text{null}(B) = \text{null}(B^T B)$

Onto matrices

$$AB = A\begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$$

$$= \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_n \end{bmatrix}$$

$Ax = y$

$\text{range}(A) = \{Ax \mid x \in \mathbb{R}^n\}$

$\text{range}(AB) = \{A(Bx) \mid x \in \mathbb{R}^m\}$

- if **range**$(A) = \mathbb{R}^m$ then A is right invertible. To see this, let b_i be such that $Ab_i = e_i$, and let $B = \left[\, b_1, \ldots, b_m \,\right]$, then $AB = I$.

- if A is right invertible, then **range** $A = \mathbb{R}^m$, because **range**$(A) \supset$ **range**(AB)

- A is left invertible iff A^T is right invertible

$\text{range}(A) \supset \text{range}(\overset{I}{AB})$

$\mathbb{R}^m = \text{range}(I) \subseteq \text{range}(A)$

$\hookrightarrow \text{range}(A) = \mathbb{R}^m$

$x = Ix$

$y \in \text{range}(AB) \Rightarrow y = ABx$

$\Rightarrow y = A(Bx)$

$\Rightarrow y \in \text{range}(A)$

Interpretations of range

suppose $v \in \textbf{range}(A), w \notin \textbf{range}(A)$

$y = Ax$ represents *measurement* of x

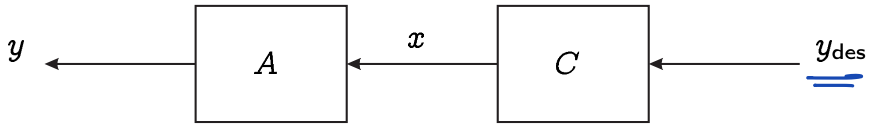
- $y = v$ is a *possible* or *consistent* sensor signal
- $y = w$ is *impossible* or *inconsistent*; sensors have failed or model is wrong

$y = Ax$ represents *output* resulting from input x

- v is a possible result or output
- w cannot be a result or output

range(A) characterizes the *possible results* or *achievable outputs*

Right invertibility and control



$$x = C y_{des}$$

$$y = Ax = \underbrace{AC}_{I} y_{des} = y_{des}$$

- apply right-inverse C at *input* of A
- then output $y = ACy_{\text{des}} = y_{\text{des}}$ as desired