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Matrix norm

- norm of a matrix

Gain of a matrix in a direction

suppose $A \in \mathbb{R}^{m \times n}$ (not necessarily square or symmetric)

for $x \in \mathbb{R}^n$, $||Ax||/||x||$ gives the *amplification factor* or *gain* of A in the direction x

obviously, gain varies with direction of input x

questions:

- what is maximum gain of A
  (and corresponding maximum gain direction)?
- what is minimum gain of A
  (and corresponding minimum gain direction)?
- how does gain of A vary with direction?

Matrix norm

the *norm* of a matrix A is

$$||A|| = \max_{x \neq 0} \frac{||Ax||}{||x||}$$

- also called the *operator norm*, *spectral norm* or *induced norm*
- gives the maximum *gain* or *amplification* of A

Matrix norm

$$\boxed{\|A\| = \sqrt{\lambda_{\max}(A^T A)}}$$

- because

$$\max_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2} = \max_{x \neq 0} \frac{x^T A^T A x}{\|x\|^2} = \lambda_{\max}(A^T A)$$

- similarly the minimum gain is given by

$$\min_{x \neq 0} \|Ax\|/\|x\| = \sqrt{\lambda_{\min}(A^T A)}$$

Input directions

note that

- $A^\mathsf{T} A \in \mathbb{R}^{n \times n}$ is symmetric and $A^\mathsf{T} A \geq 0$ so $\lambda_{\mathsf{min}},\ \lambda_{\mathsf{max}} \geq 0$
- 'max gain' input direction is $x = q_1$, eigenvector of $A^\mathsf{T} A$ associated with λ_{max}
- 'min gain' input direction is $x = q_n$, eigenvector of $A^\mathsf{T} A$ associated with λ_{min}

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad A^\mathsf{T} A = \begin{bmatrix} 35 & 44 \\ 44 & 56 \end{bmatrix}$$

$$A^\mathsf{T} A = \begin{bmatrix} 0.620 & -0.785 \\ 0.785 & 0.620 \end{bmatrix} \begin{bmatrix} 90.7 & 0 \\ 0 & 0.265 \end{bmatrix} \begin{bmatrix} 0.620 & -0.785 \\ 0.785 & 0.620 \end{bmatrix}^\mathsf{T}$$

then $\|A\| = \sqrt{\lambda_{\max}(A^\mathsf{T} A)} = 9.53$:

$$\left\| \begin{bmatrix} 0.620 \\ 0.785 \end{bmatrix} \right\| = 1, \qquad \left\| A \begin{bmatrix} 0.620 \\ 0.785 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2.19 \\ 5.00 \\ 7.81 \end{bmatrix} \right\| = 9.53$$

min gain is $\sqrt{\lambda_{\min}(A^\mathsf{T} A)} = 0.514$:

$$\left\| \begin{bmatrix} -0.785 \\ 0.620 \end{bmatrix} \right\| = 1, \qquad \left\| A \begin{bmatrix} -0.785 \\ 0.620 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0.45 \\ 0.12 \\ -0.21 \end{bmatrix} \right\| = 0.514$$

for all $x \neq 0$, we have

$$0.514 \leq \|Ax\|/\|x\| \leq 9.53$$

Properties of the matrix norm

satisfies the usual properties of a norm:

- *scaling:* $||cA|| = |c|||A||$ for $c \in \mathbb{R}$.
- *triangle inequality:* $||A + B|| \leq ||A|| + ||B||$.
- *definiteness:* $||A|| = 0 \iff A = 0$.

Properties of the matrix norm

also

- for any x, $||Ax|| \leq ||A||||x||$
- $||A|| = ||A^T||$
- consistent with vector norm: matrix norm of $a \in \mathbb{R}^{n\times 1}$ is $\sqrt{\lambda_{\max}(a^\mathsf{T} a)} = \sqrt{a^\mathsf{T} a}$
- norm of product: $||AB|| \leq ||A||||B||$
- $||A|| \geq \max\limits_i \max\limits_j |a_{ij}|$

Frobenius norm

$$||A||_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}$$

- called the *Frobenius norm*
- $||A|| \leq ||A||_F$
- $||A||_F = \big(\textbf{trace}(A^\mathsf{T}A)\big)^{\frac{1}{2}}$