

Symmetric matrices and quadratic forms

- ▶ eigenvectors of symmetric matrices
- ▶ quadratic forms
- ▶ inequalities for quadratic forms
- ▶ positive semidefinite matrices

Eigenvalues of symmetric matrices

if $A \in \mathbb{R}^{n \times n}$ is symmetric, i.e., $A = A^T$, then the eigenvalues of A are real

to see this, suppose $Av = \lambda v$, $v \neq 0$, $v \in \mathbb{C}^n$, then

$$\bar{v}^T Av = \bar{v}^T (Av) = \lambda \bar{v}^T v = \lambda \sum_{i=1}^n |v_i|^2$$

but also

$$\bar{v}^T Av = \overline{(Av)}^T v = \overline{(\lambda v)}^T v = \bar{\lambda} \sum_{i=1}^n |v_i|^2$$

so we have $\lambda = \bar{\lambda}$, i.e., $\lambda \in \mathbb{R}$ (hence, can assume $v \in \mathbb{R}^n$)

Eigenvectors of symmetric matrices

there is a set of n orthonormal eigenvectors of A

- ▶ i.e., $q_1, \dots, q_n$ s.t. $Aq_i = \lambda_i q_i$, $q_i^\top q_j = \delta_{ij}$
- ▶ in matrix form: there is an orthogonal Q s.t.

$$Q^{-1}AQ = Q^\top AQ = \Lambda$$

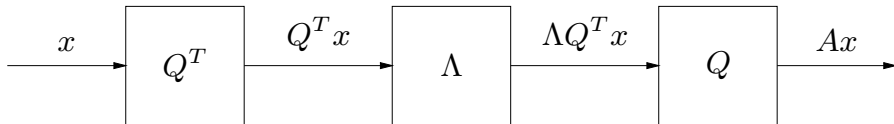
- ▶ hence we can express A as

$$A = Q\Lambda Q^\top = \sum_{i=1}^n \lambda_i q_i q_i^\top$$

- ▶ in particular, q_i are both left and right eigenvectors

Interpretations

$A = Q\Lambda Q^T$ corresponds to



linear mapping $y = Ax$ can be decomposed as

- ▶ resolve into q_i coordinates
- ▶ scale coordinates by λ_i
- ▶ reconstitute with basis q_i

Geometrical interpretation

multiplication by A is the same as

- ▶ rotate by Q^T
- ▶ diagonal real scale ('dilation') by Λ
- ▶ rotate back by Q

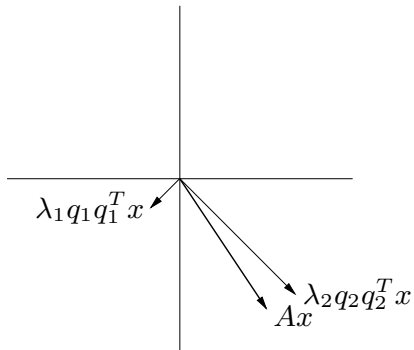
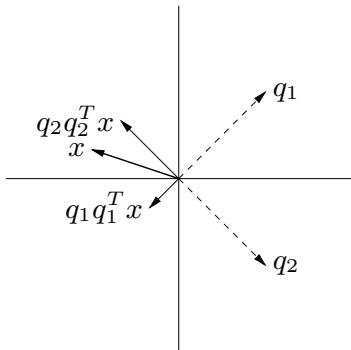
decomposition

$$A = \sum_{i=1}^n \lambda_i q_i q_i^T$$

expresses A as linear combination of 1-dimensional projections

Example:

$$\begin{aligned} A &= \begin{bmatrix} -1/2 & 3/2 \\ 3/2 & -1/2 \end{bmatrix} \\ &= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^T \end{aligned}$$



Proof

eigenvectors corresponding to distinct eigenvalues are orthogonal

- ▶ since λ_i distinct, have $v_1, \dots, v_n$, a set of linearly independent eigenvectors of A

$$Av_i = \lambda_i v_i, \quad \|v_i\| = 1$$

- ▶ then $v_i^T(Av_j) = \lambda_j v_i^T v_j = (Av_i)^T v_j = \lambda_i v_i^T v_j$
- ▶ and $(\lambda_i - \lambda_j)v_i^T v_j = 0$
- ▶ for $i \neq j$, $\lambda_i \neq \lambda_j$, hence $v_i^T v_j = 0$
- ▶ in this case we can say: eigenvectors *are* orthogonal
- ▶ in general case (λ_i not distinct) we must say: eigenvectors *can be chosen* to be orthogonal

Quadratic forms

a *quadratic form* is a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ of the form

$$f(x) = x^T A x = \sum_{i,j=1}^n A_{ij} x_i x_j$$

► in a quadratic form we may as well assume $A = A^T$ since

$$x^T A x = x^T ((A + A^T)/2) x$$

$((A + A^T)/2)$ is called the *symmetric part* of A)

► **uniqueness:** if $x^T A x = x^T B x$ for all $x \in \mathbb{R}^n$ and $A = A^T$, $B = B^T$, then $A = B$

Examples

quadratic forms

► $\|Bx\|^2 = x^\top B^\top Bx$

► $\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2$

► $\|Fx\|^2 - \|Gx\|^2$

sets defined by quadratic forms:

► $\{ x \mid f(x) = a \}$ is called a *quadratic surface*

► $\{ x \mid f(x) \leq a \}$ is called a *quadratic region*

Inequalities for quadratic forms

suppose $A = A^\top$, $A = Q\Lambda Q^\top$ with eigenvalues sorted so $\lambda_1 \geq \dots \geq \lambda_n$ then

$$x^\top Ax \leq \lambda_1 x^\top x$$

because

$$\begin{aligned} x^\top Ax &= x^\top Q\Lambda Q^\top x \\ &= (Q^\top x)^\top \Lambda (Q^\top x) \\ &= \sum_{i=1}^n \lambda_i (q_i^\top x)^2 \\ &\leq \lambda_1 \sum_{i=1}^n (q_i^\top x)^2 \\ &= \lambda_1 \|x\|^2 \end{aligned}$$

Inequalities

- ▶ similar argument shows $x^\top Ax \geq \lambda_n \|x\|^2$, so we have

$$\lambda_n x^\top x \leq x^\top Ax \leq \lambda_1 x^\top x$$

- ▶ sometimes λ_1 is called $\lambda_{\max}$, λ_n is called $\lambda_{\min}$

- ▶ note also that

$$q_1^\top A q_1 = \lambda_1 \|q_1\|^2, \quad q_n^\top A q_n = \lambda_n \|q_n\|^2,$$

so the inequalities are tight

Positive semidefinite and positive definite matrices

suppose $A = A^T \in \mathbb{R}^{n \times n}$

we say A is **positive semidefinite** if $x^T A x \geq 0$ for all x

- ▶ *this is written* $A \succeq 0$ (and sometimes $A \succeq 0$)
- ▶ $A \succeq 0$ if and only if $\lambda_{\min}(A) \geq 0$, *i.e.*, all eigenvalues are nonnegative
- ▶ **not** the same as $A_{ij} \geq 0$ for all i, j

we say A is **positive definite** if $x^T A x > 0$ for all $x \neq 0$

- ▶ denoted $A > 0$
- ▶ $A > 0$ if and only if $\lambda_{\min}(A) > 0$, *i.e.*, all eigenvalues are positive

Matrix inequalities

- ▶ we say A is *negative semidefinite* if $-A \geq 0$
- ▶ we say A is *negative definite* if $-A > 0$
- ▶ otherwise, we say A is *indefinite*

matrix inequality: if A and B are both symmetric, we use $A < B$ to mean $B - A > 0$.

- ▶ many variations, for example $A \geq B$ means $A - B \geq 0$,
- ▶ $A > B$ means $x^T A x > x^T B x$ for all $x \neq 0$

Matrix inequalities

many properties that you'd guess hold actually do, *e.g.*,

- ▶ if $A \geq B$ and $C \geq D$, then $A + C \geq B + D$
- ▶ if $B \leq 0$ then $A + B \leq A$
- ▶ if $A \geq 0$ and $\alpha \geq 0$, then $\alpha A \geq 0$
- ▶ $A^2 \geq 0$
- ▶ if $A > 0$, then $A^{-1} > 0$

matrix inequality is only a *partial order*: we can have

$$A \not\geq B, \quad B \not\geq A$$

(such matrices are called *incomparable*)