Eigenvectors and diagonalization

- eigenvectors
- diagonalization

Eigenvectors and eigenvalues

$\lambda \in \mathbb{C}$ is called an *eigenvalue* of $A \in \mathbb{C}^{n \times n}$ if

$$\boxed{\mathcal{X}(\lambda) = \mathbf{det}(\lambda I - A) = 0}$$

equivalent to:

- there exists nonzero $v \in \mathbb{C}^n$ s.t. $(\lambda I - A)v = 0$, $i.e.$,

  $$Av = \lambda v$$

  any such v is called an *eigenvector* of A (associated with eigenvalue λ)

- there exists nonzero $w \in \mathbb{C}^n$ s.t. $w^\mathsf{T}(\lambda I - A) = 0$, $i.e.$,

  $$w^\mathsf{T} A = \lambda w^\mathsf{T}$$

  any such w is called a *left eigenvector* of A

Complex eigenvalues and eigenvectors

- if v is an eigenvector of A with eigenvalue λ, then so is αv, for any $\alpha \in \mathbb{C}$, $\alpha \neq 0$
- even when A is real, eigenvalue λ and eigenvector v can be complex
- when A and λ are real, we can always find a real eigenvector v associated with λ: if $Av = \lambda v$, with $A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{R}$, and $v \in \mathbb{C}^n$, then

  $$A \Re v = \lambda \Re v, \qquad A \Im v = \lambda \Im v$$

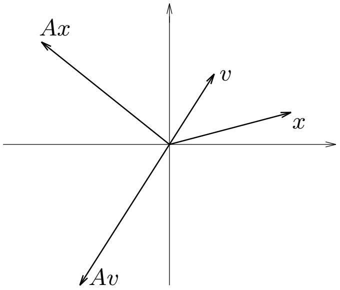
  so $\Re v$ and $\Im v$ are real eigenvectors, if they are nonzero
  (and at least one is)
- *conjugate symmetry*: if A is real and $v \in \mathbb{C}^n$ is an eigenvector associated with $\lambda \in \mathbb{C}$, then $\overline{v}$ is an eigenvector associated with $\overline{\lambda}$:

  taking conjugate of $Av = \lambda v$ we get $\overline{Av} = \overline{\lambda v}$, so

  $$A\overline{v} = \overline{\lambda}\overline{v}$$

we'll assume A is real from now on ...

Scaling interpretation



(assume $\lambda \in \mathbb{R}$ for now; we'll consider $\lambda \in \mathbb{C}$ later)

if v is an eigenvector, effect of A on v is very simple: scaling by λ

- $\lambda \in \mathbb{R}$, $\lambda > 0$: v and Av point in same direction
- $\lambda \in \mathbb{R}$, $\lambda < 0$: v and Av point in opposite directions
- $\lambda \in \mathbb{R}$, $|\lambda| < 1$: Av smaller than v
- $\lambda \in \mathbb{R}$, $|\lambda| > 1$: Av larger than v

(we'll see later how this relates to stability of continuous- and discrete-time systems. . .)

Diagonalization

suppose $v_1, \ldots, v_n$ is a *linearly independent* set of eigenvectors of $A \in \mathbb{R}^{n \times n}$:

$$Av_i = \lambda_i v_i, \quad i = 1, \ldots, n$$

express as

$$A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

define $T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ and $\Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$, so

$$\boxed{AT = T\Lambda}$$

Diagonalization

$$\boxed{T^{-1}AT = \Lambda}$$

- T invertible means $v_1, \ldots, v_n$ linearly independent
- similarity transformation by T *diagonalizes* A
- existence of invertible T such that

  $$T^{-1}AT = \Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$$

  is equivalent to existence of a linearly independent set of n eigenvectors
- we say A is *diagonalizable*
- if A is not diagonalizable, it is sometimes called *defective*

Not all matrices are diagonalizable

example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

- characteristic polynomial is $\mathcal{X}(s) = s^2$, so $\lambda = 0$ is only eigenvalue
- eigenvectors satisfy $Av = 0v = 0$, *i.e.*

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

- so all eigenvectors have form $v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$ where $v_1 \neq 0$
- thus, A cannot have two independent eigenvectors

Distinct eigenvalues

if A has distinct eigenvalues then A is diagonalizable

- distinct eigenvalues means $\lambda_i \neq \lambda_j$ for $i \neq j$
- the converse is false — A can have repeated eigenvalues but still be diagonalizable

Diagonalization and left eigenvectors

rewrite $T^{-1}AT = \Lambda$ as $T^{-1}A = \Lambda T^{-1}$, or

$$\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} A = \Lambda \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

where $w_1^T, \ldots, w_n^T$ are the rows of T^{-1}

thus

$$w_i^T A = \lambda_i w_i^T$$

$i.e.$, the rows of T^{-1} are (lin. indep.) left eigenvectors, normalized so that

$$w_i^T v_j = \delta_{ij}$$

($i.e.$, left & right eigenvectors chosen this way are *dual bases*)

Diagonalization

diagonalization simplifies many matrix expressions

powers ($i.e.$, discrete-time solution to $x(k+1) = Ax(k)$):

$$\begin{aligned} A^k &= \left(T\Lambda T^{-1}\right)^k \\ &= \left(T\Lambda T^{-1}\right) \cdots \left(T\Lambda T^{-1}\right) \\ &= T\Lambda^k T^{-1} \\ &= T\,\textbf{diag}(\lambda_1^k, \ldots, \lambda_n^k) T^{-1} \end{aligned}$$

(for $k < 0$ only if A invertible, $i.e.$, all $\lambda_i \neq 0$)

Analytic function of a matrix

for any analytic function $f : \mathbb{R} \to \mathbb{R}$, *i.e.*, given by power series

$$f(a) = \beta_0 + \beta_1 a + \beta_2 a^2 + \beta_3 a^3 + \cdots$$

we can define $f(A)$ for $A \in \mathbb{R}^{n\times n}$ (*i.e.*, overload f) as

$$f(A) = \beta_0 I + \beta_1 A + \beta_2 A^2 + \beta_3 A^3 + \cdots$$

substituting $A = T\Lambda T^{-1}$, we have

$$\begin{aligned} f(A) &= \beta_0 I + \beta_1 A + \beta_2 A^2 + \beta_3 A^3 + \cdots \\ &= \beta_0 TT^{-1} + \beta_1 T\Lambda T^{-1} + \beta_2 (T\Lambda T^{-1})^2 + \cdots \\ &= T\left(\beta_0 I + \beta_1 \Lambda + \beta_2 \Lambda^2 + \cdots\right) T^{-1} \\ &= T\,\mathbf{diag}(f(\lambda_1), \ldots, f(\lambda_n))T^{-1} \end{aligned}$$