

Range and Null Space

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Nullspace of a matrix

the *nullspace* of $A \in \mathbb{R}^{m \times n}$ is defined as

$$\text{null}(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

- ▶ $\text{null}(A)$ is set of vectors mapped to zero by $y = Ax$
- ▶ $\text{null}(A)$ is set of vectors orthogonal to all rows of A

$\text{null}(A)$ gives *ambiguity* in x given $y = Ax$:

- ▶ if $y = Ax$ and $z \in \text{null}(A)$, then $y = A(x + z)$
- ▶ conversely, if $y = Ax$ and $y = A\tilde{x}$, then $\tilde{x} = x + z$ for some $z \in \text{null}(A)$

$\text{null}(A)$ is also written $\mathcal{N}(A)$

Zero nullspace

A is called *one-to-one* if 0 is the only element of its nullspace

$$\text{null}(A) = \{0\}$$

Equivalently,

- ▶ x can always be uniquely determined from $y = Ax$
(*i.e.*, the linear transformation $y = Ax$ doesn't 'lose' information)
- ▶ mapping from x to Ax is one-to-one: different x 's map to different y 's
- ▶ columns of A are independent (hence, a basis for their span)
- ▶ A has a *left inverse*, *i.e.*, there is a matrix $B \in \mathbb{R}^{n \times m}$ s.t. $BA = I$
- ▶ $A^T A$ is invertible

Zero nullspace

- ▶ if A has a left inverse then $\text{null}(A) = \{0\}$ (proof by contradiction)
- ▶ $\text{null}(A) = \text{null}(A^T A)$
- ▶ if $\text{null}(A) = \{0\}$ then A is left invertible, because $A^T A$ is invertible, so $B = (A^T A)^{-1} A^T$ is a left inverse

Two interpretations of nullspace

suppose $z \in \text{null}(A)$, and $y = Ax$ represents *measurement* of x

- ▶ z is undetectable from sensors — get zero sensor readings
- ▶ x and $x + z$ are indistinguishable from sensors: $Ax = A(x + z)$

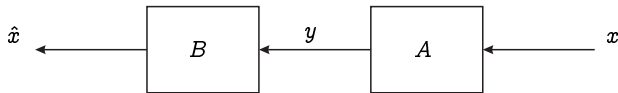
$\text{null}(A)$ characterizes *ambiguity* in x from measurement $y = Ax$

alternatively, if $y = Ax$ represents *output* resulting from input x

- ▶ z is an input with no result
- ▶ x and $x + z$ have same result

$\text{null}(A)$ characterizes *freedom of input choice* for given result

Left invertibility and estimation



- ▶ apply left-inverse B at output of A
- ▶ then estimate $\hat{x} = BAx = x$ as desired
- ▶ *non-unique*: both B and C are left inverses of A

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

Range of a matrix

the *range* of $A \in \mathbb{R}^{m \times n}$ is defined as

$$\text{range}(A) = \{Ax \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

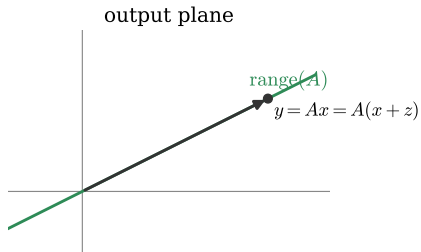
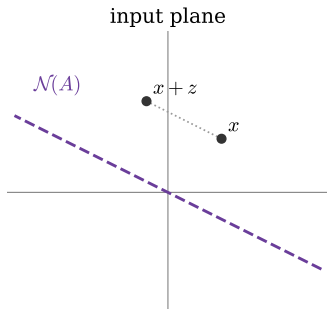
$\text{range}(A)$ can be interpreted as

- ▶ the set of vectors that can be 'hit' by linear mapping $y = Ax$
- ▶ the span of columns of A
- ▶ the set of vectors y for which $Ax = y$ has a solution

$\text{range}(A)$ is also written $\mathcal{R}(A)$

Range and nullspace: a picture

inputs x and $x + z$ with $z \in \mathbf{null}(A)$ map to the *same* output y ; the range is the span of the columns of A



Onto matrices

A is called *onto* if $\text{range}(A) = \mathbb{R}^m$

equivalently,

- ▶ $Ax = y$ can be solved in x for any y
- ▶ columns of A span $\mathbb{R}^m$
- ▶ A has a *right inverse*, i.e., there is a matrix $B \in \mathbb{R}^{n \times m}$ s.t. $AB = I$
- ▶ rows of A are independent
- ▶ $\text{null}(A^\top) = \{0\}$
- ▶ $AA^\top$ is invertible

Onto matrices

- ▶ if $\text{range}(A) = \mathbb{R}^m$ then A is right invertible. To see this, let b_i be such that $Ab_i = e_i$, and let $B = [b_1, \dots, b_m]$, then $AB = I$.
- ▶ if A is right invertible, then $\text{range } A = \mathbb{R}^m$, because $\text{range}(A) \supset \text{range}(AB)$
- ▶ A is left invertible iff A^T is right invertible

Interpretations of range

suppose $v \in \text{range}(A), w \notin \text{range}(A)$

$y = Ax$ represents *measurement* of x

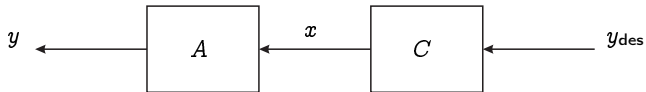
- ▶ $y = v$ is a *possible* or *consistent* sensor signal
- ▶ $y = w$ is *impossible* or *inconsistent*; sensors have failed or model is wrong

$y = Ax$ represents *output* resulting from input x

- ▶ v is a possible result or output
- ▶ w cannot be a result or output

$\text{range}(A)$ characterizes the *possible results* or *achievable outputs*

Right invertibility and control



- ▶ apply right-inverse C at *input* of A
- ▶ then output $y = ACy_{\text{des}} = y_{\text{des}}$ as desired