

Linear functions

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Linear equations

Consider system of linear equations:

$$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$\vdots$$

$$y_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n$$

Can be written in matrix form as $y = Ax$, where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Linear functions

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is *linear* if

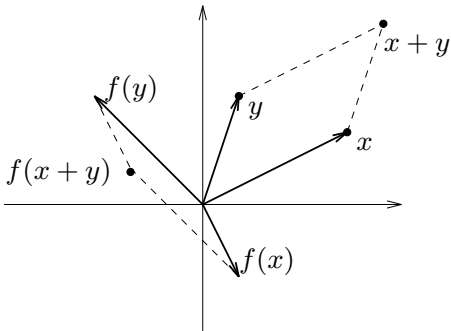
► $f(x + y) = f(x) + f(y), \forall x, y \in \mathbb{R}^n$

(additivity)

► $f(\alpha x) = \alpha f(x), \forall x \in \mathbb{R}^n \forall \alpha \in \mathbb{R}$

(homogeneity)

i.e., *superposition* holds



Exercise: linear or not?

For each f , decide whether it is linear ($a \in \mathbb{R}^n$, $b \in \mathbb{R}$ fixed):

- ▶ $f(x) = 2x_1 - 3x_2$
- ▶ $f(x) = a^\top x + b$, $b \neq 0$
- ▶ $f(x) = x_1 x_2$
- ▶ $y_i = x_1 + x_2 + \cdots + x_i$ (running sum)
- ▶ $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ rotates the plane by an angle θ
- ▶ $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ projects x onto the line $y = x$
- ▶ $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ projects x onto the line $y = 1$

Matrix multiplication function

- ▶ Consider function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $f(x) = Ax$, where $A \in \mathbb{R}^{m \times n}$
- ▶ Matrix multiplication function f is linear
- ▶ **Converse** is true: **any** linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be written as $f(x) = Ax$ for some $A \in \mathbb{R}^{m \times n}$
- ▶ Representation via matrix multiplication is unique: for any linear function f there is only one matrix A for which $f(x) = Ax$ for all x
- ▶ $y = Ax$ is a concrete representation of a generic linear function

Exercise

Show that $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear if and only if $f(x) = Ax$ for some $A \in \mathbb{R}^{m \times n}$.

Interpretations of $y = Ax$

- ▶ y is measurement or observation; x is unknown to be determined
- ▶ x is 'input' or 'action'; y is 'output' or 'result'
- ▶ $y = Ax$ defines a function or transformation that maps $x \in \mathbb{R}^n$ into $y \in \mathbb{R}^m$

Interpretation of a_{ij}

$$y_i = \sum_{j=1}^n a_{ij} x_j$$

a_{ij} is *gain factor* from j th input (x_j) to i th output (y_i)

- ▶ i th *row* of A concerns i th *output*
- ▶ j th *column* of A concerns j th *input*
- ▶ $a_{27} = 0$ means 2nd output (y_2) doesn't depend on 7th input (x_7)
- ▶ $|a_{31}| \gg |a_{3j}|$ for $j \neq 1$ means y_3 depends mainly on x_1
- ▶ $|a_{52}| \gg |a_{i2}|$ for $i \neq 5$ means x_2 affects mainly y_5
- ▶ A is lower triangular, *i.e.*, $a_{ij} = 0$ for $i < j$, means y_i only depends on $x_1, \dots, x_i$
- ▶ A is diagonal, *i.e.*, $a_{ij} = 0$ for $i \neq j$, means i th output depends only on i th input

More generally, **sparsity pattern** of A , *i.e.*, list of zero/nonzero entries of A , shows which x_j affect which y_i

Linearization

- ▶ If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at $x_0 \in \mathbb{R}^n$, then

$$x \text{ near } x_0 \implies f(x) \text{ very near } f(x_0) + Df(x_0)(x - x_0)$$

where

$$Df(x_0)_{ij} = \left. \frac{\partial f_i}{\partial x_j} \right|_{x_0}$$

is derivative (Jacobian) matrix

- ▶ With $y = f(x)$, $y_0 = f(x_0)$, define *input deviation* $\delta x := x - x_0$, *output deviation* $\delta y := y - y_0$
- ▶ Then we have $\delta y \approx Df(x_0)\delta x$
- ▶ When deviations are small, they are (approximately) related by a linear function