A Short Note on Complex Numbers

EE263: Matrix Methods

Stanford University, Summer 2026

We are about to run into complex numbers in the course, when we study eigenvalues. Even a matrix full of real numbers can have complex eigenvalues, and when it does, those numbers are telling us something concrete about the matrix: rotation, oscillation, growth, or decay. This note is a compact refresher so that the eigenvalue material lands cleanly. It is deliberately short. The goal is not a full course on complex analysis, but a working feel for what complex numbers are and for the one identity that makes them easy to reason about, $e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$.

1 Where they come from

Some equations have no real solution. The simplest is

$$x^2 + 1 = 0,$$

since $x^2 \geq 0$ for every real x. Rather than declare this equation unsolvable, we *invent* a new number whose square is -1, call it i, and agree to compute with it by the usual rules of algebra. This one new symbol turns out to be enough: once we allow i, *every* polynomial factors completely. That is a remarkable payoff for a single definition, and it is the reason complex numbers show up whenever we solve polynomial equations, which is exactly what finding eigenvalues amounts to.

$$\boxed{\mathrm{i}^2 = -1.}$$

A **complex number** is anything of the form

$$z = a + b\,\mathrm{i}, \qquad a, b \in \mathbb{R}.$$

We call $a = \operatorname{Re} z$ the **real part** and $b = \operatorname{Im} z$ the **imaginary part** (note: $\operatorname{Im} z$ is the real number b, not $b\,\mathrm{i}$). The set of all such numbers is written $\mathbb{C}$. A real number is just the special case $b = 0$, so $\mathbb{R}$ sits inside $\mathbb{C}$.

2 Arithmetic

We add and multiply complex numbers exactly as we would any algebraic expression, using $\mathrm{i}^2 = -1$ to simplify:

$$(a + b\,\mathrm{i}) + (c + d\,\mathrm{i}) = (a + c) + (b + d)\,\mathrm{i},$$

$$(a + b\,\mathrm{i})(c + d\,\mathrm{i}) = (ac - bd) + (ad + bc)\,\mathrm{i}.$$

Addition just adds the parts. Multiplication mixes them, and the $-bd$ term is where $\mathrm{i}^2 = -1$ does its work.

Conjugate. The **complex conjugate** of $z = a + b\,\mathrm{i}$ flips the sign of the imaginary part:

$$\overline{z} = a - b\,\mathrm{i}.$$

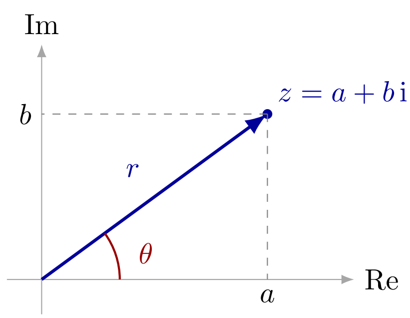
Conjugation is useful because $z\,\overline{z} = a^2 + b^2$ is always a nonnegative *real* number. It also gives a clean fact we will use constantly with eigenvalues: if a polynomial has real coefficients and z is a root, then $\overline{z}$ is a root too. Complex roots of real polynomials come in conjugate pairs. Since the eigenvalues of a real matrix are the roots of a real polynomial (its characteristic polynomial), complex eigenvalues of a real matrix always appear in conjugate pairs $\sigma \pm \mathrm{i}\omega$.

Division. To divide, multiply top and bottom by the conjugate of the denominator, which turns the denominator real:

$$\frac{1}{c + d\,\mathrm{i}} = \frac{c - d\,\mathrm{i}}{(c + d\,\mathrm{i})(c - d\,\mathrm{i})} = \frac{c - d\,\mathrm{i}}{c^2 + d^2}.$$

3 The complex plane

The pair (a, b) suggests a picture. We plot $z = a + b\,\mathrm{i}$ as the point (a, b) in the plane, with the real part on the horizontal axis and the imaginary part on the vertical axis. This is the **complex plane**. In this picture a complex number is a point, or equally a vector from the origin, and addition of complex numbers is just vector addition.



Two numbers attached to this picture matter enormously. The **modulus** (or magnitude, or absolute value) is the length of the arrow,

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z\,\overline{z}},$$

and the **argument** is the angle it makes with the positive real axis, measured counterclockwise,

$$\theta = \arg z.$$

From the right triangle, $\tan\theta = b/a$, so $\theta = \arctan(b/a)$ once you correct for the quadrant using the signs of a and b. The correction matters: $z = 1 + \mathrm{i}$ and $z = -1 - \mathrm{i}$ give the same ratio $b/a = 1$ but sit in opposite quadrants (angles $\pi/4$ and $-3\pi/4$), and arctan alone cannot tell them apart. The quadrant-aware version is common enough that most programming languages package it as a single function `atan2(b,a)`, which you may see if you compute an argument in code. Writing $r = |z|$ and reading a, b off the triangle gives the **polar form** of a complex number.

$$\boxed{z = r(\cos\theta + \mathrm{i}\sin\theta), \qquad r = |z| \geq 0.}$$

4 Why polar form is the good one

Rectangular form $a + b\,\mathrm{i}$ is convenient for adding. Polar form is convenient for *multiplying*, and multiplication is what powers of a matrix are made of. Multiply two numbers in polar form and watch what happens to the moduli and the angles:

$$\big(r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)\big)\big(r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)\big) = r_1 r_2\big(\cos(\theta_1+\theta_2) + \mathrm{i}\sin(\theta_1+\theta_2)\big),$$

using the angle-sum identities for cos and sin. In words:

$$\boxed{|z_1 z_2| = |z_1|\,|z_2|, \qquad \arg(z_1 z_2) = \arg z_1 + \arg z_2.}$$

Multiplying complex numbers *multiplies their lengths and adds their angles*. So multiplication by a fixed complex number is a scaling together with a rotation. That single sentence is the geometric heart of the whole subject, and it is exactly the behavior we will see when a real matrix has complex eigenvalues: repeated application scales by a fixed factor and rotates by a fixed angle each step.

5 Euler's formula

The angle-adding rule above should look familiar: it is how exponents behave, $e^s e^t = e^{s+t}$. That is not a coincidence. If we substitute $x = \mathrm{i}\theta$ into the power series for e^x and collect the real and imaginary terms, they regroup into exactly the series for $\cos\theta$ and $\sin\theta$:

$$e^{\mathrm{i}\theta} = \sum_{k=0}^{\infty} \frac{(\mathrm{i}\theta)^k}{k!} = \underbrace{\left(1 - \tfrac{\theta^2}{2!} + \tfrac{\theta^4}{4!} - \cdots\right)}_{\cos\theta} + \mathrm{i}\underbrace{\left(\theta - \tfrac{\theta^3}{3!} + \tfrac{\theta^5}{5!} - \cdots\right)}_{\sin\theta}.$$

This is **Euler's formula**, the single most useful identity for our purposes.

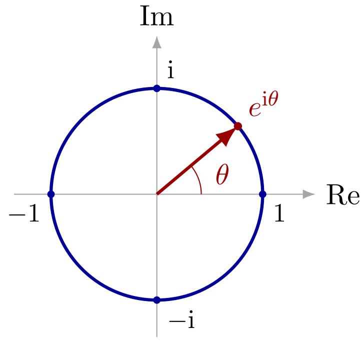
$$\boxed{e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta.}$$

With it, polar form collapses to something you can manipulate like an ordinary exponential:

$$z = r\,e^{\mathrm{i}\theta}.$$

Now every rule from the previous section is automatic. Multiplication: $r_1 e^{\mathrm{i}\theta_1}\, r_2 e^{\mathrm{i}\theta_2} = r_1 r_2\, e^{\mathrm{i}(\theta_1+\theta_2)}$, lengths multiply and angles add because exponents add. The conjugate is $\overline{z} = r\,e^{-\mathrm{i}\theta}$, a reflection across the real axis.

As θ runs from 0 to 2π, the number $e^{\mathrm{i}\theta}$ traces the **unit circle**: it always has modulus 1, and its angle is θ. So $e^{\mathrm{i}\theta}$ is "the point on the unit circle at angle θ."



A few landmark values are worth carrying in your head. They are just specific points on that circle:

$$e^{\mathrm{i}\cdot 0} = 1, \quad e^{\mathrm{i}\pi/2} = \mathrm{i}, \quad e^{\mathrm{i}\pi} = -1, \quad e^{\mathrm{i}3\pi/2} = -\mathrm{i}, \quad e^{\mathrm{i}2\pi} = 1.$$

The relation $e^{\mathrm{i}\pi} = -1$, rearranged as $e^{\mathrm{i}\pi} + 1 = 0$, is the famous one; for us it is mainly a sanity check that "rotate by π" sends 1 to -1.

Remark (Real signals). Euler's formula also lets us write the everyday sinusoids as combinations of $e^{\pm\mathrm{i}\theta}$: $\cos\theta = \frac{1}{2}\big(e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}\big)$ and $\sin\theta = \frac{1}{2\mathrm{i}}\big(e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}\big)$. A real oscillation is a conjugate pair of complex exponentials, which is another face of the "complex things come in conjugate pairs" theme.

6 Powers and roots

Powers are where polar form pays off the most, and they are precisely what we need for eigenvalues. Raising $z = re^{\mathrm{i}\theta}$ to the n-th power just raises the modulus and multiplies the angle:

$$z^n = r^n e^{\mathrm{i}n\theta} = r^n\big(\cos n\theta + \mathrm{i}\sin n\theta\big).$$

The second equality (for $r = 1$) is **De Moivre's formula**. Notice the whole behavior of z^n as n grows is decided by $r = |z|$: if $r < 1$ the powers shrink to 0, if $r > 1$ they blow up, and if $r = 1$ they stay on the unit circle and simply keep rotating. Hold onto that trichotomy.

Example 1 (The powers of a number on the unit circle)

Let $z = e^{\mathrm{i}\pi/4}$, the point at 45° with modulus 1. Then $z^2 = e^{\mathrm{i}\pi/2} = \mathrm{i}$, $z^4 = e^{\mathrm{i}\pi} = -1$, and $z^8 = e^{\mathrm{i}2\pi} = 1$. The powers march around the unit circle in 45° steps and never grow or decay. Compare $w = 0.9\,e^{\mathrm{i}\pi/4}$: it rotates the same way but each power is 0.9 times shorter, so $w^n \to 0$. Same angle, different modulus, completely different long-run behavior.

Going the other way, taking roots, the same picture explains why there are exactly n n-th roots of any nonzero complex number, spread evenly around a circle: to undo "multiply the angle by n" you may add any multiple of 2π before dividing. We will not need roots much, but it is the same modulus-and-angle bookkeeping.

7 Back to eigenvalues

Here is the payoff, stated in advance so the pieces above have a target. A real $n \times n$ matrix A can have complex eigenvalues, and by the conjugate-pair fact they come in pairs $\lambda, \overline{\lambda}$. Write such an eigenvalue in polar form,

$$\lambda = \sigma + \mathrm{i}\omega = r\,e^{\mathrm{i}\theta}, \qquad r = |\lambda|, \quad \theta = \arg\lambda.$$

Everything we care about is read off r and θ:

- The modulus r controls size. Under repeated multiplication (say the dynamics $x_{k+1} = Ax_k$), the mode attached to λ scales like r^k: it *decays* if $r < 1$, *grows* if $r > 1$, and holds steady if $r = 1$. This is the trichotomy from the previous section, applied to λ^k.
- The argument θ controls oscillation. Since $\lambda^k = r^k e^{\mathrm{i}k\theta}$, the angle advances by θ each step, so the mode rotates, *i.e.* oscillates, at that rate. A nonzero imaginary part is exactly the signature of oscillation.

The cleanest instance is a pure rotation. The matrix

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

rotates the plane by θ, and its eigenvalues are precisely $e^{\pm\mathrm{i}\theta}$: modulus 1 (a rotation changes no lengths) and angle $\pm\theta$ (the rotation rate). When you meet this next week, the complex eigenvalues will not be a curiosity. They are the natural language for saying "this matrix rotates by θ and scales by r each time you apply it."

One-page summary

- $\mathrm{i}^2 = -1$; $\quad z = a + b\,\mathrm{i}$ with $\operatorname{Re} z = a$, $\operatorname{Im} z = b$; $\quad \overline{z} = a - b\,\mathrm{i}$; $\quad |z| = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}}$.
- Polar form: $z = r\,e^{\mathrm{i}\theta}$ with $r = |z|$, $\theta = \arg z$; the point at angle θ, distance r from the origin.
- Euler: $e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$; the unit circle.
- Multiplying multiplies moduli and adds angles: it is scale + rotate.
- Powers: $z^n = r^n e^{\mathrm{i}n\theta}$; the modulus r decides growth ($r > 1$), decay ($r < 1$), or steady rotation ($r = 1$).
- Real matrices: complex eigenvalues come in conjugate pairs $\sigma \pm \mathrm{i}\omega = re^{\pm\mathrm{i}\theta}$; r is growth/decay, θ is oscillation rate.