Conditional Gaussians

<u>RV</u> X — discrete $\to$ PMF : $P_X(k) = \mathbb{P}(X=k)$

— continuous $\to$ PDF : $f_X(x)$

$P_X(x)$

$\mathbb{P}(a \le X \le b) = \int_a^b f_X(x)dx$

expected value $\mathbb{E}[X] = \sum x P_X(x) \;/\; \int x f_X(x)\,dx$

variance $Var(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2\right)$

<u>Random vectors</u> : $X \in \mathbb{R}^n \to X = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$

$$Cov(X_i, X_j) = \mathbb{E}\left((X-\mu_i)(X-\mu_j)\right)$$

$$Cov(X) = \mathbb{E}\left((X - \mathbb{E}X)(X - \mathbb{E}X)^T\right)$$

$$= \begin{array}{c|c} & j \\ \hline i & \begin{bmatrix} \\ \cdots \; Cov(X_i, X_j) \\ \vdots \end{bmatrix} \end{array} \in \mathbb{R}^{n \times n}$$

<u>Gaussians</u>:

<u>1-D</u>: $P_X(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right)$

$X \sim N(\mu, \sigma^2)$

<u>n-D</u>: $P_X(x) = \dfrac{1}{(2\pi)^{n/2}\,|\det\Sigma|^{1/2}} \exp\left(\dfrac{-1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$

$X \sim N(\underset{\mathbb{R}^n}{\mu}, \underset{\mathbb{R}^{n\times n}}{\Sigma})$ $\quad \underset{PD}{\Sigma > 0}$

Conditional pdf

- the conditional pdf $p^{|y}$ of x given $y = y_{\text{meas}}$ is

$$p^{|y}(x, y_{\text{meas}}) = \frac{p(x, y_{\text{meas}})}{p^{y}(y_{\text{meas}})}$$

Conditional mean and covariance

- the *conditional mean* of x given y is

$$\mathbf{E}(x \,|\, y = y_{\mathrm{meas}}) = \int x\, p^{|y}(x, y_{\mathrm{meas}})\, dx$$

  this is a *function of y_{meas}*

- the *conditional covariance* of x given y is

$$\mathbf{cov}(x \,|\, y = w) = \mathbf{E}\Big(\big(x - f(w)\big) \big(x - f(w)\big)^T \,\Big|\, y = w \Big)$$

  here $f(w) = \mathbf{E}(x \,|\, y = w)$. The conditional covariance is also a function of y_{meas}

Conditional pdf for a Gaussian

$$X = \begin{bmatrix} x_1 \\ \cdots \\ x_2 \end{bmatrix} \begin{matrix} \}\, n-k \\ \}\, k \end{matrix}$$

Suppose $x \sim \mathcal{N}(0, \Sigma)$, and

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

Suppose we measure $x_2 = y$. We would like to find the conditional pdf of x_1 given $x_2 = y$

- Is it Gaussian?
- What is the *conditional mean* $\mathbf{E}\big(x_1 \,|\, x_2 = y\big)$ of x_1 given x_2
- What is the *conditional covariance* $\mathbf{cov}(x_1 \,|\, x_2 = y)$ of x_1 given x_2?

Conditional pdf for a Gaussian

- the pdf of x is

$$p^x(x) = c_1 \exp\left(-\frac{1}{2}x^T\Sigma^{-1}x\right)$$

- by the completion of squares formula

$$\Sigma^{-1} = \begin{bmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{bmatrix} \begin{bmatrix} (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{bmatrix}$$

- hence

$$x^T\Sigma^{-1}x = (x_1 - Lx_2)^T T^{-1}(x_1 - Lx_2) + x_2^T\Sigma_{22}^{-1}x_2$$

where

$$T = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \qquad L = \Sigma_{12}\Sigma_{22}^{-1}$$

Conditional pdf for a Gaussian

- the conditional pdf of x_1 given x_2 is therefore

$$p^{|x_2}(x_1, y) = \frac{c_1}{p^{x_2}(y)} \exp\left(-\frac{1}{2}(x_1 - Ly)^T T^{-1}(x_1 - Ly) - \frac{1}{2} y^T \Sigma_{22}^{-1} y \right)$$

$$= \frac{c_1}{p^{x_2}(y)} \exp\left(-\frac{1}{2} y^T \Sigma_{22}^{-1} y \right) \exp\left(-\frac{1}{2}(x_1 - Ly)^T T^{-1}(x_1 - Ly) \right)$$

- therefore $p^{|x_2}(x_1, y)$ is *Gaussian*

$$p^{|x_2}(x_1, y) = c_2(y) \exp\left(-\frac{1}{2}(x_1 - Ly)^T T^{-1}(x_1 - Ly) \right)$$

where $c_2(y)$ is such that $\displaystyle\int p^{|x_2}(x_1, y)\, dx_1 = 1$

Conditional pdf for a Gaussian

$X \sim N(\mu, \Sigma^2)$, $X = \begin{bmatrix} X_1 \in \mathbb{R}^k \\ \cdots\cdots \\ X_2 \in \mathbb{R}^{n-k} \end{bmatrix}$

- If $x \sim \mathcal{N}(0, \Sigma)$, then the conditional pdf of x_1 given $x_2 = y$ is *Gaussian*
- The *conditional mean* is

$$\mathbf{E}(x_1 \,|\, x_2 = y) = \Sigma_{12}\Sigma_{22}^{-1}y$$

$\Sigma_{12} \to \rho\sigma_1\sigma_2$, $\Sigma_{22}^{-1} \to \frac{1}{\sigma_2^2}$, y

$= \rho \frac{\sigma_1}{\sigma_2} y$

it is a *linear function* of y

- the *conditional covariance* is

$$\mathbf{cov}(x_1 \,|\, x_2 = y) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$\sigma_1^2 - \rho^2\sigma_1^2\sigma_2^2 \cdot \frac{1}{\sigma_2^2}$

$= \sigma_1^2(1-\rho^2)$

it does *not* depend on y

$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

$\mathrm{Cov}(X) = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$

$\rho = \frac{\mathrm{Cov}(X_1, X_2)}{\sigma_1\sigma_2}$

Conditional pdf for a Gaussian

- Here

$$\mathbf{cov}(x) = \begin{bmatrix} 2 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$

- We have

$$L = 0.8 \qquad T = 1.36$$

- Hence

$$\mathbf{E}(x_1 \mid x_2 = 2) = 1.6$$

$$\mathbf{cov}(x_1 \mid x_2 = 2) = 0.8$$