Matrix norm

- norm of a matrix

Gain of a matrix in a direction

$$\|A\| = \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax\|}{\|x\|}$$

suppose $A \in \mathbb{R}^{m \times n}$ (not necessarily square or symmetric)

for $x \in \mathbb{R}^n$, $\|Ax\|/\|x\|$ gives the *amplification factor* or *gain* of A in the direction x

obviously, gain varies with direction of input x

$$A \in \mathbb{R}^{m \times 1} \rightarrow x \in \mathbb{R}$$

$$\|A\| = \max \frac{\|Ax\|}{\|x\|}$$

$$= \max_{x} \frac{\cancel{|x|}\,\|A\|}{\cancel{|x|}} = \|A\|$$

questions:

- what is maximum gain of A
  (and corresponding maximum gain direction)?
- what is minimum gain of A
  (and corresponding minimum gain direction)?

$$\min \frac{\|Ax\|}{\|x\|} \rightarrow 0 \rightarrow Ax = 0 \rightarrow x \in \operatorname{null}(A)$$

$$\min \frac{\|Ax\|}{\|x\|} \rightarrow \; > 0 \rightarrow \operatorname{null}(A) = \{0\}$$

- how does gain of A vary with direction?

Matrix norm

$$\|A\|^2 = \max_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2}$$

$$= \max_{x \neq 0} \frac{x^T A^T A x}{x^T x} = \lambda_{max}(A^T A)$$

the *norm* of a matrix A is

$$\boxed{\begin{gathered} B \in \mathbb{R}^{n \times n} \\ \lambda_{min} \|x\|^2 \leq x^T B x \leq \lambda_{max} \|x\|^2 \\ \lambda_{min} \leq \frac{x^T B x}{\|x\|^2} \leq \lambda_{max} \end{gathered}}$$

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

$$\Rightarrow \|A\| = \sqrt{\lambda_{max}(A^T A)}$$

- also called the *operator norm*, *spectral norm* or *induced norm*
- gives the maximum *gain* or *amplification* of A

Matrix norm

$$\|A\| = \sqrt{\lambda_{\max}(A^\top A)}$$

- because

$$\max_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2} = \max_{x \neq 0} \frac{x^\top A^\top A x}{\|x\|^2} = \lambda_{\max}(A^\top A)$$

- similarly the minimum gain is given by

$$\min_{x \neq 0} \|Ax\|/\|x\| = \sqrt{\lambda_{\min}(A^\top A)}$$

$A^T A \geq 0$

$\Updownarrow$

$\lambda_{min} \geq 0$

$\lambda_{min}(A^T A) = 0 \Rightarrow A^T A$ not full rank $\Rightarrow x \in null(A^T A)$

$\Rightarrow x \in null(A)$

Input directions

note that

- $A^\mathsf{T} A \in \mathbb{R}^{n\times n}$ is symmetric and $A^\mathsf{T} A \geq 0$ so $\lambda_{\mathsf{min}},\ \lambda_{\mathsf{max}} \geq 0$
- 'max gain' input direction is $x = q_1$, eigenvector of $A^\mathsf{T} A$ associated with λ_{max}
- 'min gain' input direction is $x = q_n$, eigenvector of $A^\mathsf{T} A$ associated with λ_{min}

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad A^\mathsf{T} A = \begin{bmatrix} 35 & 44 \\ 44 & 56 \end{bmatrix} = Q \Lambda Q^T$$

$$A^\mathsf{T} A = \begin{bmatrix} 0.620 & -0.785 \\ 0.785 & 0.620 \end{bmatrix} \begin{bmatrix} 90.7 & 0 \\ 0 & 0.265 \end{bmatrix} \begin{bmatrix} 0.620 & -0.785 \\ 0.785 & 0.620 \end{bmatrix}^\mathsf{T}$$

then $\|A\| = \sqrt{\lambda_{\max}(A^\mathsf{T} A)} = 9.53$:

$$\left\| \begin{bmatrix} 0.620 \\ 0.785 \end{bmatrix} \right\| = 1, \qquad \left\| A \begin{bmatrix} 0.620 \\ 0.785 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2.19 \\ 5.00 \\ 7.81 \end{bmatrix} \right\| = 9.53$$

min gain is $\sqrt{\lambda_{\min}(A^\mathsf{T} A)} = 0.514$:

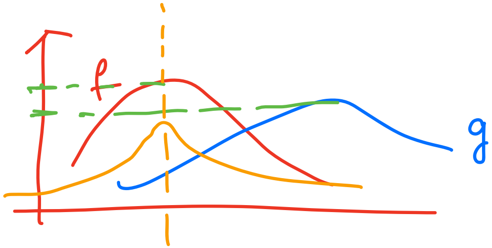
$$\left\| \begin{bmatrix} -0.785 \\ 0.620 \end{bmatrix} \right\| = 1, \qquad \left\| A \begin{bmatrix} -0.785 \\ 0.620 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0.45 \\ 0.12 \\ -0.21 \end{bmatrix} \right\| = 0.514$$

for all $x \neq 0$, we have

$$0.514 \leq \|Ax\| / \|x\| \leq 9.53$$

Properties of the matrix norm

$$\max \frac{\|cAx\|}{\|x\|} = \max |c| \frac{\|Ax\|}{\|x\|} = c\|A\|$$



satisfies the usual properties of a norm:

- *scaling:* $||cA|| = |c|||A||$ for $c \in \mathbb{R}$.
- *triangle inequality:* $||A + B|| \leq ||A|| + ||B||$.
- *definiteness:* $||A|| = 0 \iff A = 0$.

$$\hookrightarrow \max \frac{\|Ax\|}{\|x\|} = 0 \iff Ax = 0 \text{ for any } x \Rightarrow A = 0$$

$$\|u+v\| = \|u\| + \|v\| : \quad v = cu, \; c>0$$

$$\|A+B\| = \max \frac{\|(A+B)x\|}{\|x\|}$$

$$= \max \frac{\|Ax + Bx\|}{\|x\|}$$

$$\leq \max \frac{\|Ax\| + \|Bx\|}{\|x\|}$$

vector tri. ineq. $\|u+v\| \leq \|u\| + \|v\|$

$$\leq \max \frac{\|Ax\|}{\|x\|} + \max \frac{\|Bx\|}{\|x\|}$$

$\max(f+g) \leq \max f + \max g$

$$= \|A\| + \|B\|$$

T/F : u, v are max gain input directions for A, B, and we know

F u, v are not parallel. Then $\|A+B\| < \|A\| + \|B\|$.

Properties of the matrix norm

$\|A\| = \cancel{\max} \frac{\|Ax\|}{\|x\|}$ $\rightarrow$ $\|Ax\| \le \|A\| \, \|x\|$

$\ge$

$\frac{\|Ax\|}{\|x\|} \le \|A\|$

some gain < max gain

also

matrix norm

vector norm

- for any x, $||Ax|| \le \underbrace{||A||}_{\text{matrix norm}}\,\underbrace{||x||}_{\text{vector norm}}$

$\|A\|^2 = \lambda_{max}(A^TA)$

$\|A^T\|^2 = \lambda_{max}(AA^T)$

- $||A|| = ||A^T||$
- consistent with vector norm: matrix norm of $a \in \mathbb{R}^{n\times 1}$ is $\sqrt{\lambda_{\max}(a^Ta)} = \sqrt{a^Ta}$
- norm of product: $||AB|| \le ||A||||B||$ $\rightarrow$ for this to hold with equality, some max gain input for B such as x should exist, s.t. Bx is a max gain input of A.
- $||A|| \ge \max_i \max_j |a_{ij}|$

$x \longrightarrow \boxed{B} \xrightarrow{Bx} \boxed{A} \longrightarrow ABx$

Frobenius norm

$$||A||_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}$$

- called the *Frobenius norm*
- $||A|| \leq ||A||_F$
- $||A||_F = \left(\textbf{trace}(A^\top A)\right)^{\frac{1}{2}}$

$$A^T A = \begin{bmatrix} ||a_1||^2 & & \cdots \; a_i^T a_j \cdots \\ & ||a_2||^2 & \vdots \\ & \ddots & \\ & & ||a_n||^2 \end{bmatrix}$$

? eigenvalues of AB, BA?

All the non-zero eigenvalues of AB and BA are the same!

$\times B$ $\times A$

$$ABv = \lambda v \longrightarrow BA\underbrace{(BAv)}_{u} = \lambda\underbrace{(BAv)}_{u}$$

T/F Let $u, v \in \mathbb{R}^n$ s.t. $u^T v = 0$. Then the matrix

F uv^T can have exactly one non-zero eigen value.