Range and Null Space

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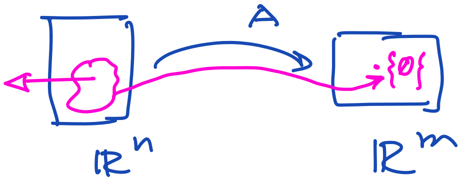
Nullspace of a matrix

$\mathrm{null}(A) = \{0\}$

null(A)



the *nullspace* of $A \in \mathbb{R}^{m \times n}$ is defined as

$$A = \begin{bmatrix} \text{—} & \tilde{a}_1^T & \text{—} \\ & \vdots & \\ \text{—} & \tilde{a}_m^T & \text{—} \end{bmatrix} \Rightarrow Ax = \begin{bmatrix} \tilde{a}_1^T x \\ \vdots \\ \tilde{a}_m^T x \end{bmatrix}$$

$$\mathbf{null}(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

- $\mathbf{null}(A)$ is set of vectors mapped to zero by $y = Ax$
- $\mathbf{null}(A)$ is set of vectors orthogonal to all rows of A

▣ null(A) is a subspace:

* $x_1, x_2 \in \mathrm{null}(A) \Rightarrow Ax_1 = Ax_2 = 0$

$\Rightarrow A(x_1 + x_2) = 0 \rightarrow x_1 + x_2 \in \mathrm{null}(A)$

* same for αx

$\mathbf{null}(A)$ gives *ambiguity* in x given $y = Ax$:

- if $y = Ax$ and $z \in \mathbf{null}(A)$, then $y = A(x+z) = \overbrace{Ax}^{y} + \overbrace{Az}^{0}$
- conversely, if $y = Ax$ and $y = A\tilde{x}$, then $\tilde{x} = x + z$ for some $z \in \mathbf{null}(A)$

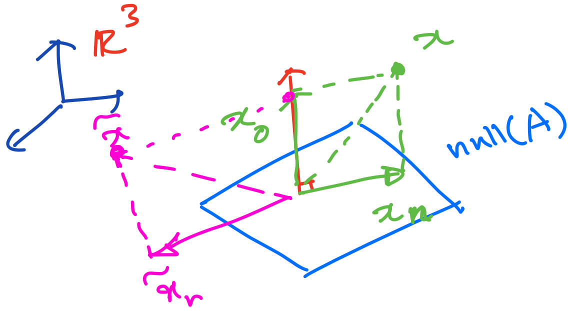
$\tilde{x} = x + z \Rightarrow z = \tilde{x} - x$

$Ax = y$

$A\tilde{x} = y$

$A\underbrace{(x - \tilde{x})}_{z} = 0$

$\mathbf{null}(A)$ is also written $\mathcal{N}(A)$



$$x = (x_n + x_0)$$

$$Ax = \underbrace{Ax_n}_{\emptyset} + Ax_0$$

$$= Ax_0$$

Zero nullspace

$y = Ax_1 \quad , \; y = Ax_2$

$0 = A(x_1 - x_2) \rightarrow x_1 - x_2 \in \text{null}(A)$

$\Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

A is called *one-to-one* if 0 is the only element of its nullspace

$$\boxed{\textbf{null}(A) = \{0\}}$$

if $z \neq 0$ exists, s.t. $Az = 0$

$\Leftrightarrow a_1 z_1 + \cdots + a_n z_n = 0$

a_i's are dependent

Equivalently,

- x can always be uniquely determined from $y = Ax$ ($i.e.$, the linear transformation $y = Ax$ doesn't 'lose' information)
- mapping from x to Ax is one-to-one: different x's map to different y's
- columns of A are independent (hence, a basis for their span)
- A has a *left inverse*, $i.e.$, there is a matrix $B \in \mathbb{R}^{n \times m}$ s.t. $BA = I$
- $A^\top A$ is invertible

$\begin{bmatrix} B \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} I \end{bmatrix}$

$\text{null}(A) = \{0\} \iff$ A has a left inverse (i.e. $B \in \mathbb{R}^{n\times m}$ exists s.t. $BA = I$)

$\Leftarrow$ $\quad Ax = 0 \Rightarrow \underbrace{BA}_{I}x = 0 \Rightarrow x = 0 \Rightarrow \text{null}(A) = 0$

$\Rightarrow$ $\quad B = (\underbrace{A^TA}_{\in\mathbb{R}^{m\times m}})^{-1}A^T \Rightarrow BA = (A^TA)^{-1}(A^TA) = I$

now I have to prove $(A^TA)^{-1}$ exists

$\text{null}(A) = 0 \iff A^TA \in \mathbb{R}^{m\times m}$ is invertible

$\boxed{\text{null}(A^TA) \overset{\subseteq}{\underset{\supseteq}{=}} \text{null}(A)}$

$\supseteq$ $\quad x \in \text{null}(A) \Rightarrow Ax = 0 \Rightarrow A^TAx = 0 \Rightarrow x \in \text{null}(A^TA)$ ✓

$\subseteq$ $\quad x \in \text{null}(A^TA) \Rightarrow A^TAx = 0$

$\Rightarrow x^TA^TAx = 0$

$\Rightarrow (Ax)^T(Ax) = 0$

$u^Tu = \|u\|^2 \qquad \Rightarrow \|Ax\|^2 = 0$

$\Rightarrow Ax = 0 \Rightarrow x \in \text{null}(A)$

Usually, when trying to prove $S = T$, where S, T are sets, we take 2 steps:

① $S \subseteq T \rightsquigarrow x \in S \Rightarrow \dots \Rightarrow x \in T$

② $T \subseteq S$

$Ax \neq 0$

$BAx = 0$

$Ax \neq 0$

$A^TAx \neq$

$A \in \mathbb{R}^{m\times n}$, $\text{null}(A) = 0$

$A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_m \\ | & & | \end{bmatrix}$, $a_m \in \mathbb{R}^n$, m

$\underline{Q1}$: compare m, n: $m < n$ ✓, $m = n$ ✓, $m > n$ ✗

$[B]_{n\times m}[A]_{m\times n} = [I]_{n\times n}$

$[A][B] = [I]_{m\times m}$

$m \neq n$

$\underline{Q_2}$: $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$: left inverse: $BA = I_{m\times m}$

What can we say about $AB \overset{?}{=} I_{n\times n}$ — always / maybe / never

Zero nullspace

- if A has a left inverse then $\textbf{null}(A) = \{0\}$ (proof by contradiction)
- $\textbf{null}(A) = \textbf{null}(A^\mathsf{T} A)$
- if $\textbf{null}(A) = \{0\}$ then A is left invertible, because $A^\mathsf{T} A$ is invertible, so $B = (A^\mathsf{T} A)^{-1} A^\mathsf{T}$ is a left inverse

Two interpretations of nullspace

suppose $z \in \textbf{null}(A)$, and $y = Ax$ represents *measurement* of x

- z is undetectable from sensors — get zero sensor readings
- x and $x + z$ are indistinguishable from sensors: $Ax = A(x + z)$

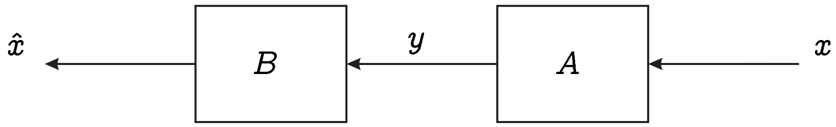
$\textbf{null}(A)$ characterizes *ambiguity* in x from measurement $y = Ax$

alternatively, if $y = Ax$ represents *output* resulting from input x

- z is an input with no result
- x and $x + z$ have same result

$\textbf{null}(A)$ characterizes *freedom of input choice* for given result

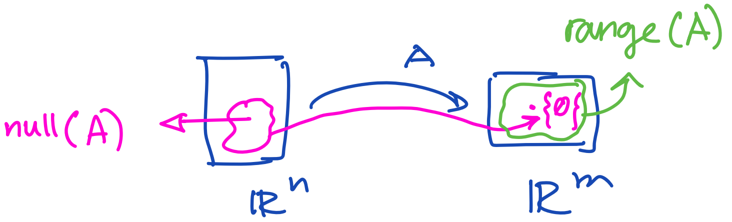
Left invertibility and estimation



- apply left-inverse B at output of A
- then estimate $\hat{x} = BAx = x$ as desired
- *non-unique:* both B and C are left inverses of A

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

Range of a matrix



the *range* of $A \in \mathbb{R}^{m \times n}$ is defined as

dim

$$\boxed{\dim(\text{range}(A)) = \text{rank}(A)}$$

$$\textbf{range}(A) = \{Ax \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

$$\text{range}(A) = \text{span}(\{a_1, \dots, a_n\})$$

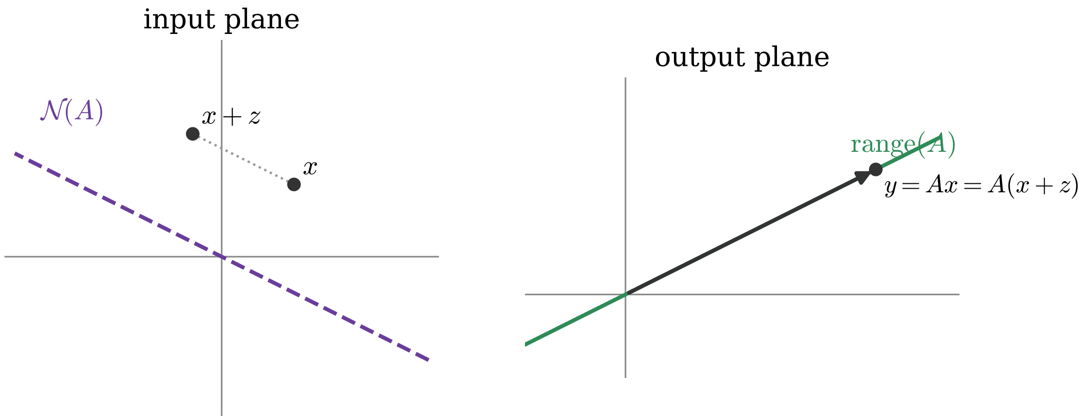
range(A) can be interpreted as

- the set of vectors that can be 'hit' by linear mapping $y = Ax$
- the span of columns of A
- the set of vectors y for which $Ax = y$ has a solution

range(A) is also written $\mathcal{R}(A)$

Range and nullspace: a picture

inputs x and $x + z$ with $z \in$ **null**(A) map to the *same* output y; the range is the span of the columns of A



Onto matrices

* onto → right inverse

$Ax = e_1 \to x = b_1$

$Ax = e_2 \to x = b_2$

$\vdots$

$B = \begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix}$

$AB = \begin{bmatrix} Ab_1 & \cdots & Ab_m \end{bmatrix} = I$

A is called *onto* if $\underbrace{\textbf{range}(A) = \mathbb{R}^m}$

↳ $\text{rank}(A) = m$

equivalently,

- $Ax = y$ can be solved in x for any y
- columns of A span $\mathbb{R}^m$
- A has a *right inverse*, *i.e.*, there is a matrix $B \in \mathbb{R}^{n \times m}$ s.t. $AB = I$
- rows of A are independent
- $\textbf{null}(A^\mathsf{T}) = \{0\}$
- AA^T is invertible

* right inverse → onto

$AB = I$ $\Big\{$ $Ax = y$ always has a solution

$x = ? \Rightarrow x = By$

$\Rightarrow Ax = A(By) = y$

$AB = I \Rightarrow B^T A^T = I$

$\Rightarrow A^T$ has a left inverse

$\Rightarrow$ columns of A^T are indep.

$\Rightarrow$ rows of A are indep.

Onto matrices

$y \in \text{range}(AB) \Rightarrow y = ABx = A(\underbrace{Bx}_{z}) \Rightarrow y \in \text{range}(A)$

- if **range**$(A) = \mathbb{R}^m$ then A is right invertible. To see this, let b_i be such that $Ab_i = e_i$, and let $B = \left[\, b_1, \ldots, b_m \,\right]$, then $AB = I$.
- if A is right invertible, then **range** $A = \mathbb{R}^m$, because **range**$(A) \supset$ **range**$(\underset{\uparrow\, I}{AB}) = \mathbb{R}^m$
- A is left invertible iff A^{T} is right invertible

Interpretations of range

suppose $v \in$ **range**(A),$w \notin$ **range**(A)

$y = Ax$ represents *measurement* of x

- $y = v$ is a *possible* or *consistent* sensor signal
- $y = w$ is *impossible* or *inconsistent*; sensors have failed or model is wrong

$y = Ax$ represents *output* resulting from input x

- v is a possible result or output
- w cannot be a result or output

range(A) characterizes the *possible results* or *achievable outputs*

Example

$A \in \mathbb{R}^{20 \times 10}$

$y = Ax$

20 measurements

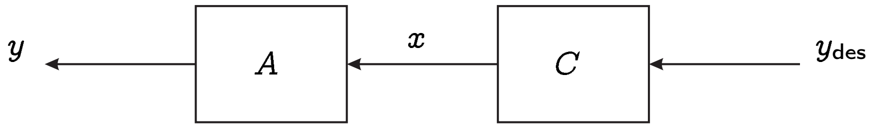
$y_1, \ldots, y_{20}$

Exactly 1 sensor is failing

how do we find it?

1. try to solve $y = Ax$
   $\Rightarrow$ we find no x
2. remove row i and solve the remaining system $\rightarrow$ when consistent, we have found the failing sensor

Right invertibility and control



$x = C y_{des}$

- apply right-inverse C at *input* of A
- then output $y = ACy_{\text{des}} = y_{\text{des}}$ as desired

one-to-one	onto
$\text{null}(A) = \{0\}$	$\text{range}(A) = \mathbb{R}^m$
indep. cols	indep. rows
left inverse	right inverse
A^TA invertible	AA^T invertible