7.1080. Hovercraft with limited range. We have a hovercraft moving in the plane with two thrusters, each pointing through the center of mass, exerting forces in the x and y directions with 100% efficiency. The hovercraft has mass 1. The discretized equations of motion for the hovercraft are

\[
x(t + 1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}
\]

where \(x_1\) and \(x_2\) are the position and velocity in the x-direction, and \(x_3\), \(x_4\) are the position and velocity in the y-direction. Here

\[
u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}
\]

is the force acting on the hovercraft for time in the interval \([t, t + 1]\). Let the position of the vehicle at time \(t\) be \(q(t) \in \mathbb{R}^2\).

a) The hovercraft starts at the origin. We’d like to apply thrust to make it move through points \(p_1, p_2, p_3\) at times \(t_1, t_2, t_3\), where

\[
p_1 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \quad p_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad p_3 = \begin{bmatrix} -\frac{3}{2} \\ 0 \end{bmatrix}
\]

\(t_1 = 6\) \quad \(t_2 = 40\) \quad \(t_3 = 50\)

We will run the hovercraft on the time interval \([0, 70]\). We’d like to apply a sequence of inputs \(u(0), u(1), \ldots, u(70)\) to make the hovercraft position pass through the above sequence of points at the specified times.

We would like to find the sequence of inputs that drives the hovercraft through the desired points which has the minimum cost, given by the sum of the squares of the forces:

\[
\sum_{t=0}^{70} \|u(t)\|^2
\]

To do this, pick \(A_{hov}\) and \(y_{des}\) to set this problem up as an equivalent minimum-norm problem, where we would like to find the minimum-norm \(u_{seq}\) which satisfies

\[
A_{hov}u_{seq} = y_{des}
\]

where \(u_{seq}\) is the sequence of force inputs

\[
u_{seq} = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(70) \end{bmatrix}
\]
Plot the trajectory of the hovercraft using this input, and the way-points $p_1, \ldots, p_3$. Also plot the optimal $u$ against time.

b) Now we would like to compute the trade-off curve between the accuracy with which the mass passes through the waypoints and the norm of the force used. Let our two objective functions be

$$J_1 = \sum_{i=1}^{3} \|q(t_i) - p_i\|^2 = \|A_{hov}u_{seq} - y_{des}\|^2$$

and

$$J_2 = \sum_{t=0}^{70} ||u(t)||^2$$

By minimizing the weighted sum

$$J_1 + \mu J_2$$

for a range of values of $\mu$, plot the trade-off curve of $J_1$ against $J_2$ showing the achievable performance. This above trade-off curve shows how we can trade-off between how accurately the hovercraft passes through the waypoints and how much input energy is used.

c) For each of the following values of $\mu$

$$\{ 10^p \mid p = -2, 0, 2, \ldots, 10 \}$$

plot the trajectories all on the same plot, together with the waypoints.

d) Now suppose we are controlling the hovercraft by radio control, and the maximum range possible between the transmitter and receiver is 2 (in whatever units we are using for distance.) Notice that, if we use the minimum-norm input then the hovercraft passes out of range, both when making its first turn and on the final stretch (between times 50 and 70).

We’d like to do something about this, but trading off the input norm as above doesn’t do the right thing; if $\mu$ is large then the hovercraft stays within range, but misses the waypoints entirely; if $\mu$ is small then it comes close to the waypoints, but goes out of range. Notice that this is particularly a problem on the final stretch between times 50 and 70; explain why this is.

e) One remedy for this problem is to solve a constrained multiobjective least-squares problem. We would like to impose the constraint that

$$A_{hov}u_{seq} = y_{des}$$

that is, achieve zero waypoint error $J_1 = 0$. We can attempt to keep the hovercraft in range by trading off the sum of the squares of the position

$$J_3 = \sum_{t=0}^{70} ||q(t)||^2$$
against input cost $J_2$ subject to this constraint. To do this, we’ll solve

$$\begin{align*}
\text{minimize} & \quad J_3 + \gamma J_2 \\
\text{subject to} & \quad A_{\text{hov}} u_{\text{seq}} = y_{\text{des}}
\end{align*}$$

First, find the matrix $W$ so that the cost function is given by

$$J_3 + \gamma J_2 = \|W u_{\text{seq}}\|^2$$

f) Now we have a problem of the form

$$\begin{align*}
\text{minimize} & \quad \|W u\|^2 \\
\text{subject to} & \quad Au = y_{\text{des}}
\end{align*}$$

This is called a \textit{weighted minimum-norm solution}; the only difference from the usual minimum-norm solution to $Au = y_{\text{des}}$ is the presence of the matrix $W$, and when $W = I$ the optimal $u$ is just given by $u_{\text{opt}} = A^\dagger y_{\text{des}}$. Show that the solution for general $W$ is

$$u_{\text{opt}} = \Sigma^{-1} A^T (A \Sigma^{-1} A^T)^{-1} y_{\text{des}}$$

where $\Sigma = W^T W$. (One way to do this is using Lagrange multipliers.) Use this to solve the remaining parts of this problem.

g) For each of the following values of $\gamma$

$$\{ 10^p \mid p = 0, 2, 4, \ldots, 20 \}$$

Plot the trajectories all on the same plot, together with the waypoints. Explain what you see.

h) By trying different values of $\gamma$, you should be able to find a trajectory which just keeps the hovercraft within range. Plot the trajectory of the hovercraft; what is the corresponding value of $\gamma$? Is this the smallest-norm input $u$ that just keeps the hovercraft within range, and drives the hovercraft through the waypoints? Explain why, or why not.

i) For a range of values of $\gamma$, plot the trade-off curve of $J_3$ against $J_2$ showing the achievable performance.

\section*{9.1410. Invariance of the unit square.} Consider the linear dynamical system $\dot{x} = Ax$ with $A \in \mathbb{R}^{2 \times 2}$. The unit square in $\mathbb{R}^2$ is defined by

$$S = \{ x \mid -1 \leq x_1 \leq 1, \quad -1 \leq x_2 \leq 1 \}.$$ 

a) Find the exact conditions on $A$ for which the unit square $S$ is invariant under $\dot{x} = Ax$. Give the conditions as explicitly as possible.

b) Consider the following statement: if the eigenvalues of $A$ are real and negative, then $S$ is invariant under $\dot{x} = Ax$. Either show that this is true, or give an explicit counterexample.
11.1820. Optimal control for maximum asymptotic growth. We consider the controllable linear system

\[ x(t + 1) = Ax(t) + Bu(t), \quad x(0) = 0, \]

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \). You can assume that \( A \) is diagonalizable, and that it has a single dominant eigenvalue (which here, means that there is one eigenvalue with largest magnitude). An input \( u(0), \ldots, u(T - 1) \) is applied over time period \( 0, 1, \ldots, T - 1 \); for \( t \geq T \), we have \( u(t) = 0 \). The input is subject to a total energy constraint:

\[ \|u(0)\|^2 + \cdots + \|u(T - 1)\|^2 \leq 1. \]

The goal is to choose the inputs \( u(0), \ldots, u(T - 1) \) that maximize the norm of the state for large \( t \). To be more precise, we’re searching for \( u(0), \ldots, u(T - 1) \), that satisfies the total energy constraint, satisfies \( \|x(t)\| \geq \|\hat{x}(t)\| \) for \( t \) large enough. Explain how to do this. You can use any of the ideas from the class, e.g., eigenvector decomposition, SVD, pseudo-inverse, etc. Be sure to summarize your final description of how to solve the problem. Unless you have to, you should not use limits in your solution. For example you cannot explain how to make \( \|x(t)\| \) as large as possible (for a specific value of \( t \)), and then say, “Take the limit as \( t \to \infty \)” or “Now take \( t \) to be really large”.

12.1930. Properties of trajectories. For each of the following statements, give the exact (necessary and sufficient) conditions on \( A \in \mathbb{R}^{n \times n} \) under which the statement holds.

a) Every trajectory of \( \dot{x} = Ax \) converges as \( t \to \infty \). This means that, for any \( x(0) \), \( x(t) \) converges to some value, which need not be zero (and can depend on \( x(0) \) and \( A \)).

b) Every trajectory of \( \dot{x} = Ax \) is bounded. This means that, for any \( x(0) \), there is an \( M \) (that can depend on \( x(0) \) and \( A \)) for which \( \|x(t)\| \leq M \) for all \( t \geq 0 \).

Your answers can refer to any concepts used in the course (eigenvalues, singular values, Jordan form, least-squares, range, nullspace, . . .). We will deduct points from answers that are technically correct, but more complicated than they need to be. You may not make any assumptions about \( A \) (e.g., that it is nonsingular, diagonalizable, etc.).

Please give only your final answer; we do not want any justification or discussion. Your answers should have a form similar to “The property in part (a) occurs if and only if all singular values of \( A \) are less than one, and \( A \) has no real eigenvalues”. (This is not the correct answer; it is only as an example of what your answer should look like.)


\[ x(t + 1) = Ax(t) + Bu(t), \]

\[ y(t) = Cx(t), \]

with \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n} \). In output feedback control we use an input which is a linear function of the output, that is,

\[ u(t) = Ky(t), \]

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where $K \in \mathbb{R}^{m \times p}$ is the *feedback gain matrix*. The resulting state trajectory is identical to that of an autonomous system,

$$x(t + 1) = \bar{A}x(t).$$

a) Write $\bar{A}$ explicitly in terms of $A$, $B$, $C$, and $K$.

b) Consider the single-input, single-output system with

$$A = \begin{bmatrix} 0.5 & 1.0 & 0.1 \\ -0.1 & 0.5 & -0.1 \\ 0.2 & 0.0 & 0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}. $$

In this case, the feedback gain matrix $K$ is a scalar (which we call simply the *feedback gain*.) The question is: find the feedback gain $K_{\text{opt}}$ such that the feedback system is maximally damped. By maximally damped, we mean that the state goes to zero with the fastest asymptotic decay rate (measured for an initial state $x(0)$ with non-zero coefficient in the slowest mode.) *Hint:* You are only required to give your answer $K_{\text{opt}}$ up to a precision of $\pm 0.01$, and you can assume that $K_{\text{opt}} \in [-2, 2]$. 

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