EE263 Homework 6

Summer 2024

7.1080. Hovercraft with limited range. We have a hovercraft moving in the plane with two thrusters, each pointing through the center of mass, exerting forces in the **x** and **y** directions with 100% efficiency. The hovercraft has mass 1. The discretized equations of motion for the hovercraft are

$$x(t+1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

where x_1 and x_2 are the position and velocity in the **x**-direction, and x_3, x_4 are the position and velocity in the **y**-direction. Here

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

is the force acting on the hovercraft for time in the interval $[t, t+1)$. Let the position of the vehicle at time t be $q(t) \in \mathbb{R}^2$.

a) The hovercraft starts at the origin. We'd like to apply thrust to make it move through points p_1, p_2, p_3 at times t_1, t_2, t_3, where

$$p_1 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \qquad p_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad p_3 = \begin{bmatrix} -\frac{3}{2} \\ 0 \end{bmatrix}$$

$$t_1 = 6 \qquad t_2 = 40 \qquad t_3 = 50$$

We will run the hovercraft on the time interval $[0, 70]$. We'd like to apply a sequence of inputs $u(0), u(1), \ldots, u(70)$ to make the hovercraft position pass through the above sequence of points at the specified times.

We would like to find the sequence of inputs that drives the hovercraft through the desired points which has the minimum cost, given by the sum of the squares of the forces:

$$\sum_{t=0}^{70} \|u(t)\|^2$$

To do this, pick A_{hov} and y_{des} to set this problem up as an equivalent minimum-norm problem, where we would like to find the minimum-norm u_{seq} which satisfies

$$A_{\text{hov}} u_{\text{seq}} = y_{\text{des}}$$

where u_{seq} is the sequence of force inputs

$$u_{\text{seq}} = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(70) \end{bmatrix}$$

Plot the trajectory of the hovercraft using this input, and the way-points $p_1, \ldots, p_3$. Also plot the optimal u against time.

b) Now we would like to compute the trade-off curve between the accuracy with which the mass passes through the waypoints and the norm of the force used. Let our two objective functions be

$$J_1 = \sum_{i=1}^{3} \|q(t_i) - p_i\|^2 = \|A_{\text{hov}} u_{\text{seq}} - y_{\text{des}}\|^2$$

and

$$J_2 = \sum_{t=0}^{70} \|u(t)\|^2$$

By minimizing the weighted sum

$$J_1 + \mu J_2$$

for a range of values of μ, plot the trade-off curve of J_1 against J_2 showing the achievable performance. This above trade-off curve shows how we can trade-off between how accurately the hovercraft passes through the waypoints and how much input energy is used.

c) For each of the following values of μ

$$\{\, 10^{\frac{p}{2}} \mid p = -2, 0, 2, \ldots, 10 \,\}$$

plot the trajectories all on the same plot, together with the waypoints.

d) Now suppose we are controlling the hovercraft by radio control, and the maximum range possible between the transmitter and receiver is 2 (in whatever units we are using for distance.) Notice that, if we use the minimum-norm input then the hovercraft passes out of range, both when making its first turn and on the final stretch (between times 50 and 70).

We'd like to do something about this, but trading off the input norm as above doesn't do the right thing; if μ is large then the hovercraft stays within range, but misses the waypoints entirely; if μ is small then it comes close to the waypoints, but goes out of range. Notice that this is particularly a problem on the final stretch between times 50 and 70; explain why this is.

e) One remedy for this problem is to solve a *constrained multiobjective least-squares* problem. We would like to impose the constraint that

$$A_{\text{hov}} u_{\text{seq}} = y_{\text{des}}$$

that is, achieve zero waypoint error $J_1 = 0$. We can attempt to keep the hovercraft in range by trading off the sum of the squares of the *position*

$$J_3 = \sum_{t=0}^{70} \|q(t)\|^2$$

against input cost J_2 subject to this constraint. To do this, we'll solve

$$\begin{array}{ll} \text{minimize} & J_3 + \gamma J_2 \\ \text{subject to} & A_{\text{hov}} u_{\text{seq}} = y_{\text{des}} \end{array}$$

First, find the matrix W so that the cost function is given by

$$J_3 + \gamma J_2 = \|W u_{\text{seq}}\|^2$$

f) Now we have a problem of the form

$$\begin{array}{ll} \text{minimize} & \|Wu\|^2 \\ \text{subject to} & Au = y_{\text{des}} \end{array}$$

This is called a *weighted minimum-norm solution*; the only difference from the usual minimum-norm solution to $Au = y_{\text{des}}$ is the presence of the matrix W, and when $W = I$ the optimal u is just given by $u_{\text{opt}} = A^{\dagger} y_{\text{des}}$. Show that the solution for general W is

$$u_{\text{opt}} = \Sigma^{-1} A^T (A \Sigma^{-1} A^T)^{-1} y_{\text{des}}$$

where $\Sigma = W^T W$. (One way to do this is using Lagrange multipliers.) Use this to solve the remaining parts of this problem.

g) For each of the following values of γ

$$\{\, 10^{\frac{p}{2}} \mid p = 0, 2, 4, \ldots, 20 \,\}$$

Plot the trajectories all on the same plot, together with the waypoints. Explain what you see.

h) By trying different values of γ, you should be able to find a trajectory which just keeps the hovercraft within range. Plot the trajectory of the hovercraft; what is the corresponding value of γ? Is this the smallest-norm input u that just keeps the hovercraft within range, and drives the hovercraft through the waypoints? Explain why, or why not.

i) For a range of values of γ, plot the trade-off curve of J_3 against J_2 showing the achievable performance.

Solution.

a) Setting

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

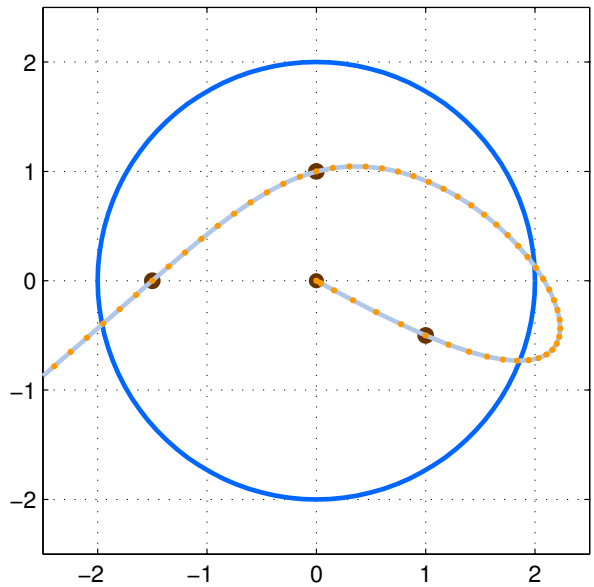
gives the position of the hovercraft at time t as

$$y(t) = \sum_{\tau=0}^{t-1} C A^{t-1-\tau} B u(\tau)$$

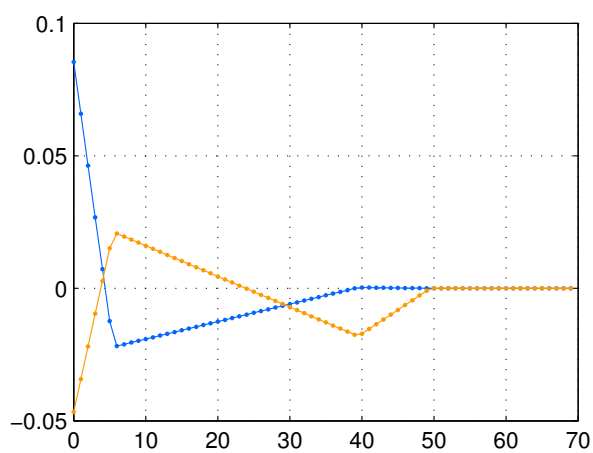
The parameters for the least-squares problem are therefore

$$A_{\text{hov}} = \begin{bmatrix} CA^{t_1-1}B & CA^{t_1-1} & \cdots & CB & 0 & 0 & \ldots & 0 \\ CA^{t_2-1}B & CA^{t_2-2}B & & & \cdots & & & 0 \\ CA^{t_3-1}B & CA^{t_3-2}B & & & \cdots & & & 0 \end{bmatrix} \qquad y_{\text{des}} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Solving this least squares problem gives optimal trajectory



The corresponding optimal input sequence is below.



b) The weighted sum objective is

$$J_1 + \mu J_2 = \left\| \begin{bmatrix} A_{\text{hov}} \\ \sqrt{\mu} I \end{bmatrix} u_{\text{seq}} - \begin{bmatrix} y_{\text{des}} \\ 0 \end{bmatrix} \right\|^2$$

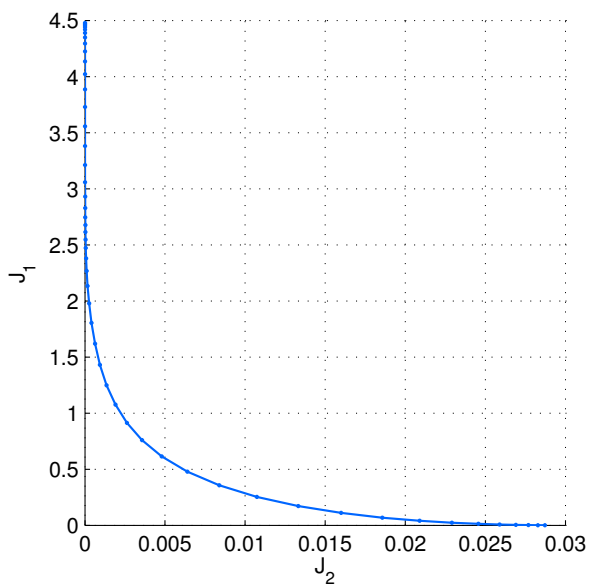
where

$$u_{\text{seq}} = \begin{bmatrix} u(0) \\ \vdots \\ u(69) \end{bmatrix}$$

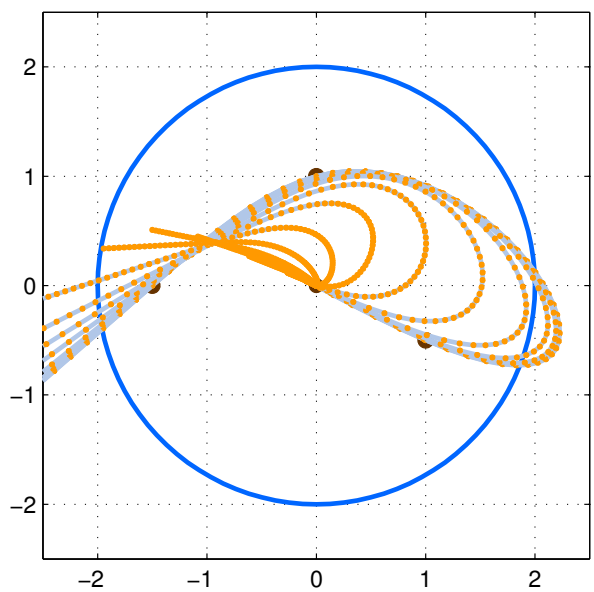
and so the optimal input sequence is given by

$$u_{\text{seq}} = \begin{bmatrix} A_{\text{way}} \\ \sqrt{\mu} I \end{bmatrix}^{\dagger} \begin{bmatrix} y_{\text{des}} \\ 0 \end{bmatrix}$$

Choosing values of μ between 1 and 10^7 using `mus=logspace(0,7,50)`, the trade-off curve is shown below.



c) All of the trajectories together are



We can see clearly that increasing μ reduces the accuracy with which the trajectory passes through the waypoints.

d) On the final stretch the input is zero, and so is unaffected by increasing μ. We were attempting to use the heuristic 'keeping u small keeps x small' but this fails, because when $u = 0$ the hovercraft just keeps going in a straight line.

e) We would like to minimize $J_3 + \gamma J_2$ subject to the constraints that the hovercraft moves through the waypoints. Denote the sequence of positions of the hovercraft by

$$y_{\text{seq}} = \begin{bmatrix} y(0) \\ \vdots \\ y(T) \end{bmatrix}$$

where $T = 70$. Then we have

$$y_{\text{seq}} = T u_{\text{seq}}$$

where T is the Toeplitz matrix

$$T = \begin{bmatrix} 0 & & & \\ CB & 0 & & \\ CAB & CB & 0 & \\ \vdots & & \ddots & \\ CA^{T-1}B & CA^{T-2}B & \dots & CB \end{bmatrix}$$

Now the cost function is

$$\begin{aligned} J_3 + \gamma J_2 &= \|T u_{\text{seq}}\|^2 + \gamma \|u_{\text{seq}}\|^2 \\ &= \|W u_{\text{seq}}\|^2 \end{aligned}$$

where

$$W = \begin{bmatrix} T \\ \sqrt{\gamma} I \end{bmatrix}$$

f) We'd like to solve

$$\begin{array}{ll} \text{minimize} & \|Wu\|^2 \\ \text{subject to} & Au = y_{\text{des}} \end{array}$$

One way to solve this is using Lagrange multipliers; if we augment the cost function by the Lagrange multipliers multiplied by the constraints, we have

$$L(u, \lambda) = u^T \Sigma u + \lambda^T (Au - y_{\text{des}})$$

and the optimality conditions are

$$\frac{\partial L}{\partial u} = 2u_{\text{opt}}^T \Sigma + \lambda^T A = 0$$

$$\frac{\partial L}{\partial \lambda} = u_{\text{opt}}^T A^T - y_{\text{des}}^T = 0$$

The first condition gives

$$u_{\text{opt}} = -\frac{1}{2} \Sigma^{-1} A^T \lambda$$

and substituting this into the second we have

$$-\frac{1}{2} A \Sigma^{-1} A^T \lambda = y_{\text{des}}$$

hence

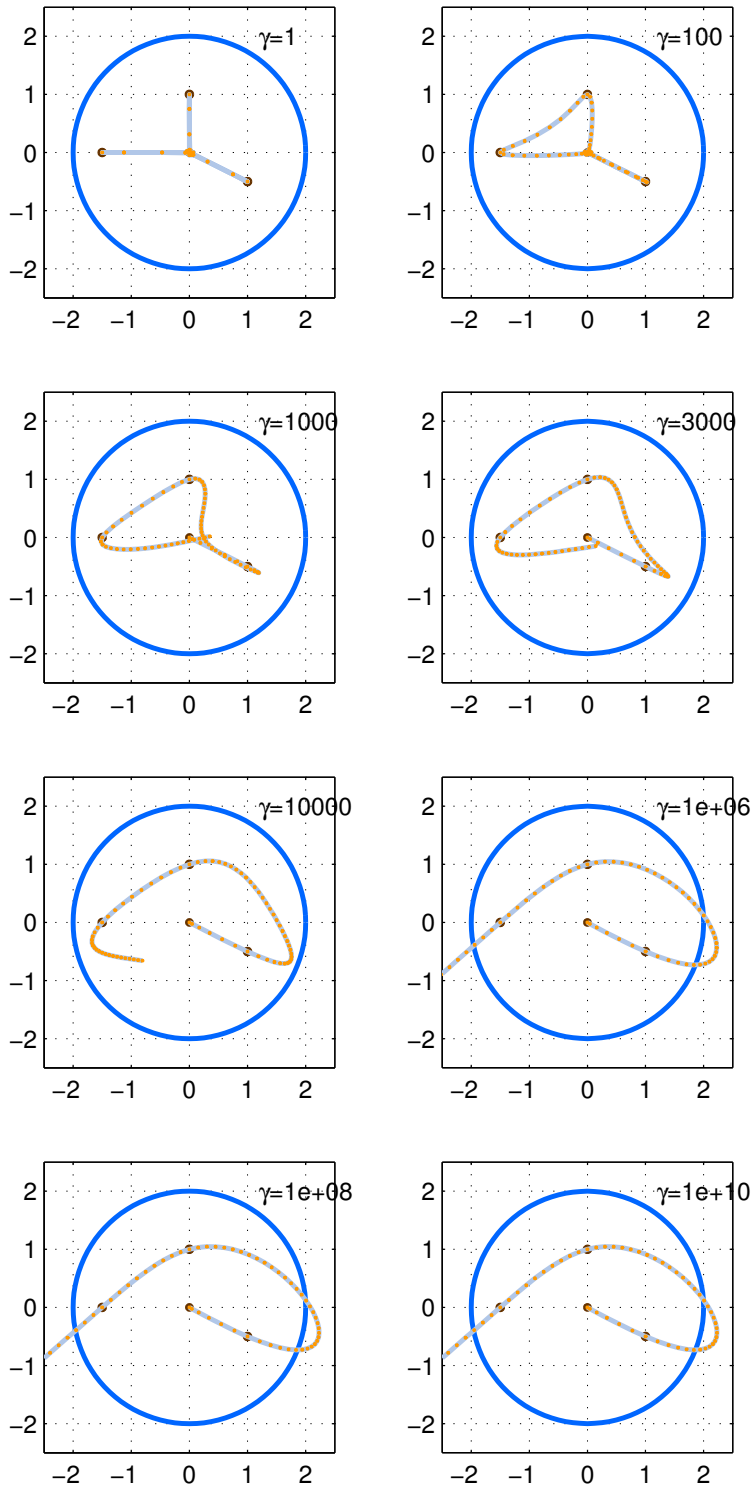
$$\lambda = -2(A\Sigma^{-1}A^T)^{-1} y_{\text{des}}$$

and

$$u_{\text{opt}} = \Sigma^{-1} A^T (A\Sigma^{-1}A^T)^{-1} y_{\text{des}}$$

as desired.

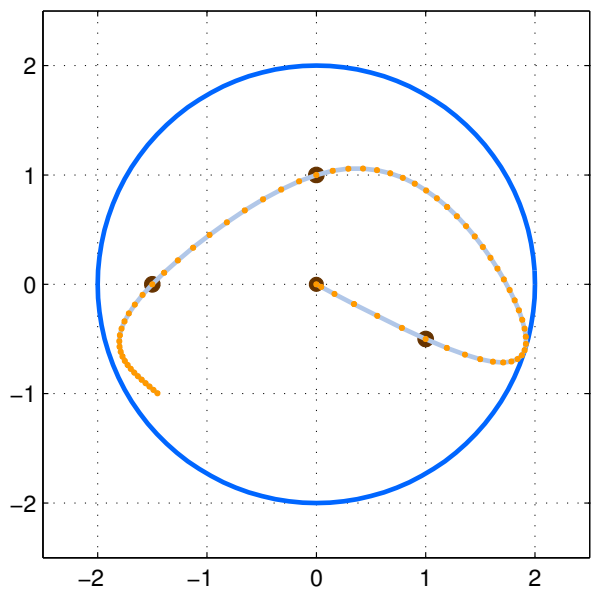
g) The trajectory for a range of γ values is shown below. (Actually these are clearer on separate plots)



We can see the trade-off clearly; decreasing γ causes the hovercraft to try very hard to

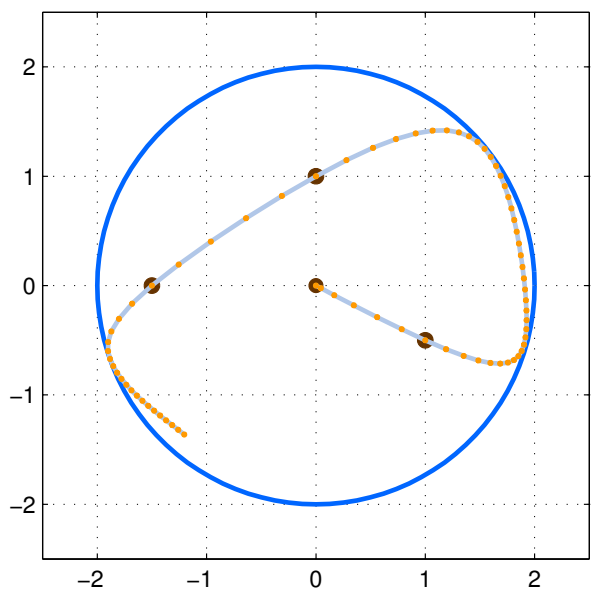
stay close to the origin. Also notice the asymmetry caused by the different times at which the hovercraft must be at the waypoints.

h) A good choice of gamma is about 1.7×10^4. Here the trajectory just remains within range, as shown below.

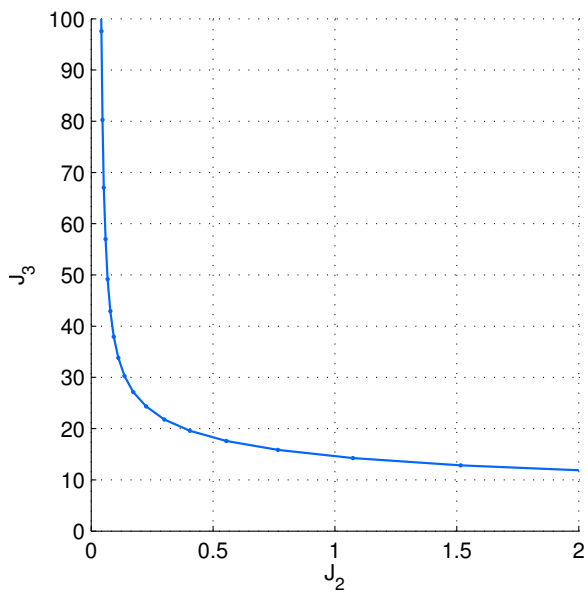


This is not the smallest-norm u that keeps the hovercraft within range and drives the hovercraft through the waypoints, because we are minimizing the *sum* of the squares of $\|q(t)\|$, rather than constraining each $\|q(t)\|$ independently. You can see this in the plot, since in the final stretch the hovercraft is expending extra effort to stay well within range, and this excessive input could be reduced.

In fact, one can compute the exact optimal, but this is not required and not covered in this course; (an approximation of) it is below.



i) The trade-off is below.



Notice that the vertical asymptote occurs when $J_2 \approx 0.03$; this is the minimum-norm of u which drives the hovercraft through the desired trajectory, as seen in part (b).

code that solves this problem
helper functions

```
function y=vec(x)
% VEC produces a vector of length m*n from an m by n matrix
```

```
%
%  function y=vec(x)
%
% Given an m by n matrix x, y=vec(x) constructs a vector y
% consisting of the columns of x stacked on top of each other
%

[m,n] = size(x);

y = reshape(x,m*n,1);


function T=sys_toeplitz(A,B,C,D,out_times,in_times);
% SYS_TOEPLITZ computes toeplitz matrices for a discrete-time LDS
%
%   T=sys_toeplitz(A,B,C,D,out_times,in_times);
%
% A,B,C,D specify a discrete-time state-space realization
%
% out_times and in_times are row vectors
%
% for example, if out_times is [1,2,4] and in_times is [0:10]
% then T is the matrix which maps
%
%   [u(0); u(1); ...  u(10)]  to  [y(1); y(2); y(4)]
%
% Notice that T is Toeplitz if out_times and in_times are
% both of the form a:b

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% setup parameters

% number of states
n=size(A,1);

% num inputs and outputs
ny=size(C,1);
nu=size(B,2);


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% construct the Toeplitz matrix

% this is neither efficient nor numerically reliable for big matrices
% but is simple and works well for small cases
```

```
T=[];
for r=out_times

  % now create a row for this output time
  T_row=[];

  for s=in_times
    % three cases; either CA^tB, D, or 0

    if s<r
      % below the diagonal
      this_block= C*A^(r-s-1)*B;

    elseif s==r
      % on the diagonal
      this_block=D;

    else
      % above the diagonal
      this_block=zeros(ny,nu);

    end

    T_row=[T_row, this_block];
  end

  T=[T; T_row];

end
```

main code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% compute min-norm input that drives a hovercraft through a
% given set of waypoints


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% parameters

% desired radius
r_max=2;

% desired time steps and positions
```

```
way_times=[ 6 40 50 ];

way_points=[    1,   0,   -1.5  ;
     -0.5,   1,    0    ];

% final time step
t_max=70;

% sampling time
h=1;

% discrete-time system
A=[1 h 0 0 ;
   0 1 0 0 ;
   0 0 1 h ;
   0 0 0 1 ];

B=[h^2/2 0;
   h 0;
   0 h^2/2 ;
   0 h ];

C=[ 1 0 0 0 ;
    0 0 1 0 ];

D=[ 0 0 ;
    0 0 ];

% number of states
n=size(A,1);

% num inputs and outputs
ny=size(C,1);
nu=size(B,2);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% real work is here

% toeplitz matrix mapping inputs to position at way_times
A_hov=sys_toeplitz(A,B,C,D, way_times, 0:t_max-1);

% find minimum norm input
u_tmp=pinv(A_hov)*vec(way_points);

% for convenience of simulation, reshape the inputs
```

```
% so that u_opt(:,k+1) is the input vector at time k
u_opt=reshape(u_tmp,2,t_max);

% simulate
x=zeros(n,t_max+1);
for k=0:t_max-1
  x(:,k+2)=A*x(:,k+1) + B*u_opt(:,k+1);
end
y=C*x;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% everything from here on is just plotting

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% plot trajectory in the plane

figure(1);
clf;
hold on;
axis equal;
axis([-2.5,2.5,-2.5,2.5]);
grid;
box on;


% plot radio range
[xs,ys]=ellipse(r_max^2*eye(2),[0;0]);
plot(xs,ys,'r');

% plot way points
for k=1:size(way_times,2)
  plot(way_points(1,k),way_points(2,k),'ko');
end

% plot trajectory
plot(y(1,:),y(2,:),'.-');



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% plot thruster input versus time

figure(2);
clf;
hold on;
```

```
grid;

h=plot((0:t_max-1),u_opt(1,1:t_max),'b.-');
h=plot((0:t_max-1),u_opt(2,1:t_max),'r.-');




%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% for a hovercraft
% compute the trade-off curve of input norm to distance norm
% subject to the constraint that
% the trajectory passes through the waypoints

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% parameters

% desired radius
r_max=2;

% desired time steps and positions
way_times=[ 6 40 50 ];

way_points=[    1,   0,   -1.5  ;
     -0.5,   1,    0    ];

n_way_points=size(way_points,2);

% final time step
t_max=70;

% sampling time
h=1;

% discrete-time system

A=[1 h 0 0 ;
   0 1 0 0 ;
   0 0 1 h ;
   0 0 0 1 ];

B=[h^2/2 0;
   h 0;
   0 h^2/2 ;
   0 h ];
```

```
C=[ 1 0 0 0 ;
    0 0 1 0 ];

D=[ 0 0 ;
    0 0 ];

% number of states
n=size(A,1);

% num inputs and outputs
ny=size(C,1);
nu=size(B,2);


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% compute trade-off for weighted min-norm problem


% toeplitz matrix mapping inputs to position at way_times
A_hov=sys_toeplitz(A,B,C,D, way_times, 0:t_max-1);

% toeplitz matrix mapping inputs to sequence of positions
A_pos=sys_toeplitz(A,B,C,D, 0:t_max, 0:t_max-1);

% desired gamma values
gammas=logspace(0,5,40);


% space to save costs
J2=[];
J3=[];

for i=1:size(gammas,2)
  this_gamma=gammas(i);

  % weight parameter - sigma inverse
  sig_inv=inv(A_pos'*A_pos + this_gamma*eye(t_max*nu));

  % stack up the way_points
  y_des=reshape(way_points,2*n_way_points,1);

  % compute input that minimizes weighted cost
  lambda=(A_hov*sig_inv*A_hov')\y_des;
  u=sig_inv*A_hov'*lambda;
```

```
  % compute corresponding trajectory in the plane
  y=A_pos*u;

  % keep track of achieved costs
  J3(i) = norm(y)^2;
  J2(i) = norm(u)^2;

  % store a nicely shaped y for plotting
  y_keep{i}=reshape(y,2,t_max+1);

end


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% everything from here is just plotting

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% first plot trade off curve

% figure setup
figure(1);
clf;
hold on;
grid;
axis([0,3,0,200]);

% plot
plot(J2,J3,'.-');
xlabel('J_2 position cost');
ylabel('J_3 input cost');
title('trade off curve')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% now plot all the different inputs

% loop over each gamma value
for i=1:size(gammas,2)

  figure(2);
  clf;
  hold on;
  axis equal;
  axis([-2.5,2.5,-2.5,2.5]);
  grid;
```

```
  box on;
  title(sprintf('gamma=%g',gammas(i)));

  % plot radio range
  [xs,ys]=ellipse(r_max^2*eye(2),[0;0]);
  plot(xs,ys,'r');

  % plot way points
  for k=1:size(way_times,2)
    plot(way_points(1,k),way_points(2,k),'ko');
  end

  % plot trajectory
  y=y_keep{i};
  plot(y(1,:),y(2,:),'.-');

  % wait for 100 milliseconds between plots
  pause(0.1);

end
```

8.1110. Simultaneous left inverse of two matrices. Consider a system where

$$y = Gx, \quad \tilde{y} = \tilde{G}x$$

where $G \in \mathbb{R}^{m\times n}$, $\tilde{G} \in \mathbb{R}^{m\times n}$. Here x is some variable we wish to estimate or find, y gives the measurements with some set of (linear) sensors, and $\tilde{y}$ gives the measurements with some *alternate* set of (linear) sensors. We want to find a *reconstruction matrix* $H \in \mathbb{R}^{n\times m}$ such that $HG = H\tilde{G} = I$. Such a reconstruction matrix has the nice property that it recovers x perfectly from *either* set of measurements (y or $\tilde{y}$), *i.e.*, $x = Hy = H\tilde{y}$. Consider the specific case

$$G = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 4 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} -3 & -1 \\ -1 & 0 \\ 2 & -3 \\ -1 & -3 \\ 1 & 2 \end{bmatrix}.$$

Either find an explicit reconstruction matrix H, or explain why there is no such H.

Solution. The requirements $HG = I$ and $H\tilde{G} = I$ are a set of linear equations in the variables H_{ij}. Since $H \in \mathbb{R}^{n\times m}$ there are mn unknowns; each equation of the form $HG = I$ gives n^2 equations, so all together we have $2n^2$ equations. Roughly speaking, it's reasonable to expect a solution to exist when there are more variables than equations, *i.e.*, $mn \geq 2n^2$, which implies that $m \geq 2n$. This condition makes sense: to invert two different sensor measurements we need a redundancy factor of two. Now let's look at the specific case given. Suppose that

$$H = \begin{bmatrix} h_1^\mathsf{T} \\ h_2^2 \end{bmatrix}$$

where $h_1, h_2 \in \mathbb{R}^5$. $HG = I$ implies that $h_1^\mathsf{T} G = e_1^\mathsf{T}$ and $h_2^\mathsf{T} G = e_2^\mathsf{T}$ where e_1 and e_2 are the unit vectors in $\mathbb{R}^2$. Similarly we should have $h_1^\mathsf{T} \tilde{G} = e_1^\mathsf{T}$ and $h_2^\mathsf{T} \tilde{G} = e_2^\mathsf{T}$. In block matrix form

$$\begin{bmatrix} G^\mathsf{T} & 0 \\ \tilde{G}^\mathsf{T} & 0 \\ 0 & G^\mathsf{T} \\ 0 & \tilde{G}^\mathsf{T} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_1 \\ e_2 \\ e_2 \end{bmatrix}.$$

Now by defining

$$A = \begin{bmatrix} G^\mathsf{T} & 0 \\ \tilde{G}^\mathsf{T} & 0 \\ 0 & G^\mathsf{T} \\ 0 & \tilde{G} \end{bmatrix}, \quad x = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \quad b = \begin{bmatrix} e_1 \\ e_2 \\ e_1 \\ e_2 \end{bmatrix}$$

we get the standard form $Ax = b$. If $b \in \text{range}(A)$ a solution exists and H can be found. In this case, A happens to be full rank so $(AA^\mathsf{T})^{-1}$ exists and we can set $x = A^\mathsf{T}(AA^\mathsf{T})^{-1}b$.

Here is the Julia code to solve this problem:

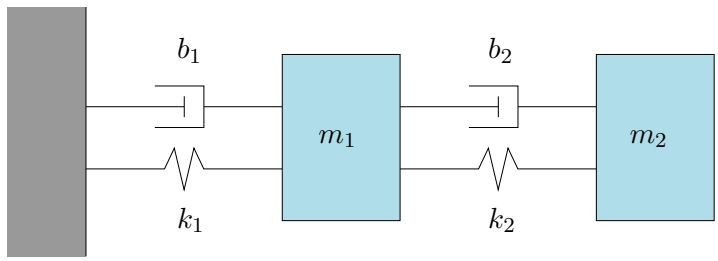
```
using LinearAlgebra;

G = [2 3; 1 0; 0 4; 1 1; -1 2]
G_tilde = [-3 -1; -1 0; 2 -3; -1 -3; 1 2]
m, n = size(G)

# Try to find simultaneous left inverse for G and G_tilde
A = [G' zeros(n, m);
     G_tilde' zeros(n, m);
     zeros(n, m) G';
     zeros(n, m) G_tilde']
b = [I(n)[:, 1]; I(n)[:, 1]; I(n)[:, 2]; I(n)[:, 2]]

if rank([A b]) != rank(A)
    print("There is no simultaneous left inverse.\n")
else
    x = A \ b
    H = [x[begin:m]';
         x[(m + 1):end]']
    print("The simultaneous left inverse is given by:\n")
    display(H)

    # Confirm that H is a simulatenous left inverse
    print("HG = I: ", all(abs.(I(n) - H * G) .< 1e-14), "\n")
    print("HG_tilde = I: ", all(abs.(I(n) - H * G_tilde) .< 1e-14), "\n")
end
```

Using this code, you can see that G and $\tilde{G}$ *do* have a simulatenous left inverse H; the specific solution H that we found running the code above is given by

$$H = \begin{bmatrix} -0.278166 & 2.09441 & 0.860917 & -1.22844 & -0.690365 \\ 0.161616 & 0.227273 & 0.0808081 & -0.30303 & 0.247475 \end{bmatrix}. \tag{1}$$

Of course, there are other solutions as well. Indeed, since there are 10 variables and 8 independent equations, there is a 2 dimensional set of H's that satisfy the required condition.

9.1490. System in a box. You are given a mysterious box containing two unit masses connected via springs and dampers as follows.



The equations of motion for this system are

$$\begin{aligned} \ddot{q}_1 &= -k_1 q_1 + k_2(q_2 - q_1) - b_1\dot{q}_1 + b_2(\dot{q}_2 - \dot{q}_1) \\ \ddot{q}_2 &= -k_2(q_2 - q_1) - b_2(\dot{q}_2 - \dot{q}_1) \end{aligned}$$

where $k_i > 0$ are spring constants, $b_i > 0$ are damping constants, and q_i is the displacement of mass i. Both masses are $m_i = 1$.

a) This mysterious box can be modeled as a continuous-time linear dynamical system

$$\dot{x} = Ax$$

Find A in terms of k_1, k_2, b_1, b_2. Use state $x = (q_1, q_2, \dot{q}_1, \dot{q}_2)$.

b) Use the forward Euler discretization

$$x(t+h) = (I + hA)x(t)$$

to simulate this system, with parameters $k_1 = 1$, $k_2 = 2$, $b_1 = 1$, $b_2 = 3$. Set the initial conditions so that $q_1(0) = 1$, $q_2(0) = 2$, and $\dot{q}_1(0) = \dot{q}_2(0) = 0$. Use time step $h = 0.1$, and simulate on time interval $[0, T]$ with $T = 15$. Plot q_1, q_2, $\dot{q}_1$, and $\dot{q}_2$ (on one plot) as functions of time.

c) Unfortunately your dog ate the documentation for this system in a box so you do not know the parameters k_1, k_2, b_1, b_2. However, we have experimental data, consisting of measurements of $x(t)$ with sample period $h = 0.1$ on time interval $[0, T]$ where $T = 15$. These may be found in the file `box.json`. We will use this data to estimate k_1, k_2, b_1, b_2.

To do this, we will find the matrix A that minimizes

$$\sum_{k=0}^{N-1} \|(I + hA)x(kh) - x(kh + h)\|^2$$

where $N = T/h$, and x is the given data. Explain how you would do this.

d) Apply your method from part (**??**) to estimate A, and report the estimate you find.

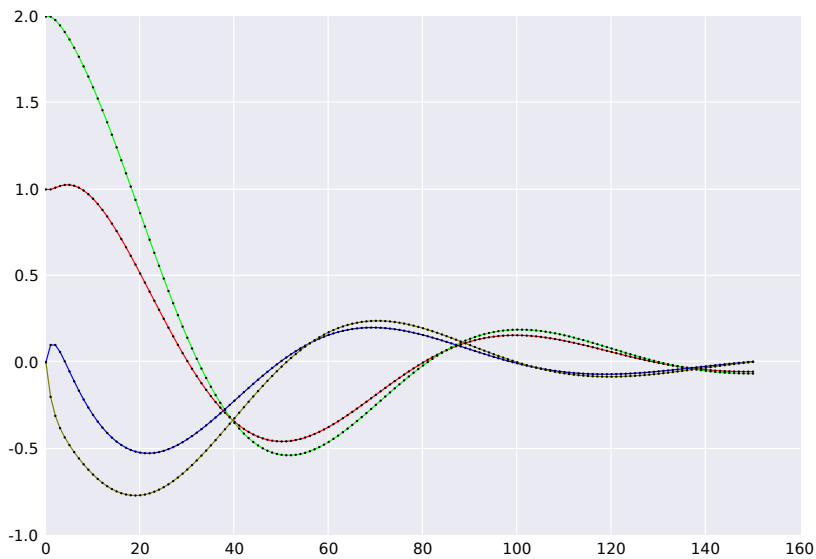
e) Using initial conditions of $x(0)$ in the dataset, and again with $h = 0.1$ and $T = 15$, simulate the system using your estimate of A and plot q_1, q_2, $\dot{q}_1$, and $\dot{q}_2$ (on one plot) as functions of time.

Solution.

a) We have

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1 + k_2) & k_2 & -(b_1 + b_2) & b_2 \\ k_2 & -k_2 & b_2 & -b_2 \end{bmatrix}$$

b) The plot is



c) The problem is equivalent to

$$\min_a . \ \|Wa - v\|$$

where

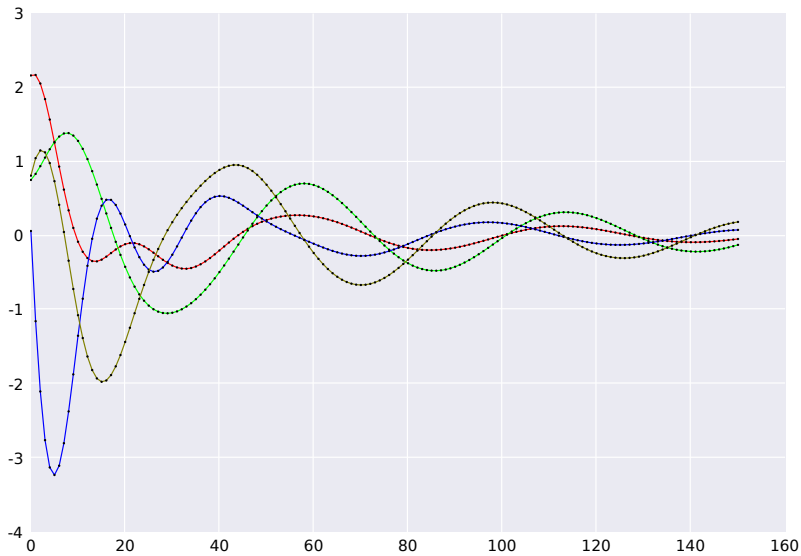
$$W = \begin{bmatrix} hx(0)^\mathsf{T} \otimes I \\ hx(h)^\mathsf{T} \otimes I \\ \vdots \\ hx((N-1)h)^\mathsf{T} \otimes I \end{bmatrix} \qquad v = \begin{bmatrix} x(h) - x(0) \\ x(2h) - x(h) \\ \vdots \\ x(Nh) - x((N-1)h) \end{bmatrix}$$

and $a = \mathrm{vec}(A)$.

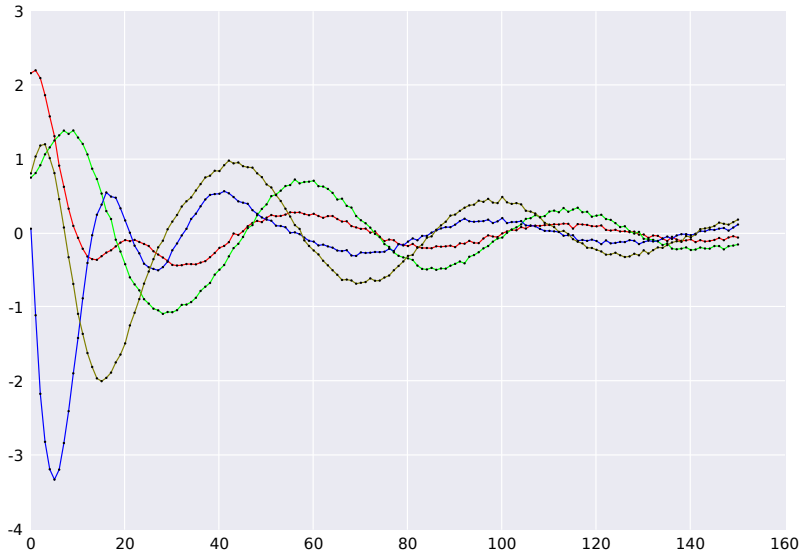
d) We find

$$A = \begin{bmatrix} -0.0120 & 0.0229 & 1.0004 & 0.0081 \\ -0.0327 & -0.0157 & -0.0175 & 1.0054 \\ -6.5739 & 2.0700 & -1.9523 & 0.7532 \\ 2.1230 & -2.0799 & 0.8136 & -0.7946 \end{bmatrix}$$

e) The plot is below.



For comparison, the original data is here also.



13.1970. Affine dynamical systems. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *affine* if it is a linear function plus a constant, *i.e.*, of the form $f(x) = Ax + b$. Affine functions are more general than linear functions, which result when $b = 0$. We can generalize linear dynamical systems to *affine dynamical systems*, which have the form

$$\dot{x} = Ax + Bu + f, \quad y = Cx + Du + g.$$

Fortunately we don't need a whole new theory for (or course on) affine systems; a simple shift of coordinates converts it to a linear dynamical system. Assuming A is invertible, define $\tilde{x} = x + A^{-1}f$ and $\tilde{y} = y - g + CA^{-1}f$. Show that $\tilde{x}$, u, and $\tilde{y}$ are the state, input, and output of a linear dynamical system.

Solution. All we have to do is to show that $\tilde{x}$, u and $\tilde{y}$ satisfy a linear dynamical system. First note that

$$\frac{d\tilde{x}}{dt} = \frac{d}{dt}(x + A^{-1}f) = \frac{dx}{dt} \quad (A^{-1}f \in \mathbb{R}^n \text{ is constant})$$

and therefore

$$\dot{\tilde{x}} = Ax + Bu + f = A(x + A^{-1}f) + Bu = A\tilde{x} + Bu.$$

Now, substituting in $y = Cx + Du + g$ for y and x in terms of $\tilde{y}$ and $\tilde{x}$ gives

$$\tilde{y} + g - CA^{-1}f = C(\tilde{x} - A^{-1}f) + Du + g = C\tilde{x} - CA^{-1}f + Du + g$$

or

$$\tilde{y} = C\tilde{x} + Du.$$

Hence, $\tilde{x}$, u and $\tilde{y}$ satisfy the following LDS:

$$\dot{\tilde{x}} = A\tilde{x} + Bu, \quad \tilde{y} = C\tilde{x} + Du.$$

18.2890. Linear dynamical systems for portfolio management. We consider a portfolio of n financial assets (like stocks) and cash, which we manage over T time steps of unit length (e.g. one month). We call $x_t \in \mathbb{R}^{n+1}$ for $t = 1, \ldots, T$ our state vector. The first n elements are our positions in each of the assets, in dollars, and the last element is the dollar amount of cash we hold. Every element of x can be either positive (for long positions) and negative (for short or borrowing). For $t = 1, \ldots, T-1$, the transition from x_t to x_{t+1} is composed of two steps.

- First, the portfolio positions change value because of market returns. Let $\mu \in \mathbb{R}^n_{++}$ be the vector of returns, where $\mathbb{R}^n_{++}$ is the set of all vectors of length n with strictly positive entries. We define the post-return portfolio $\tilde{x}_t$ to be

$$(\tilde{x}_t)_i = \begin{cases} \mu_i (x_t)_i & i = 1, \ldots, n, \\ (x_t)_i & i = n+1. \end{cases}$$

  (Intuitively, cash is unchanged, and the asset positions are multiplied by the corresponding element of the vector of returns.) For simplicity we assume that the vector of returns does not change in time.

- Then we trade. We can exchange any amount of cash for the corresponding amount of any of the assets. Note that the *only* valid trades are cash for asset. If you wish to trade some amount of an asset with the same amount of another asset, you have to perform *two trades*: trade the first asset with cash, and then trade cash with the second asset. (Think carefully about this definition of trade when you formulate the transaction costs.) For example, if we buy $c > 0$ dollars of the first asset and sell $d > 0$ dollars of the second asset the state evolves as

$$x_{t+1} = \tilde{x}_t + \begin{bmatrix} c \\ -d \\ 0 \\ \vdots \\ 0 \\ -(c-d) \end{bmatrix}.$$

Finally, we define the portfolio value $v_t \in \mathbb{R}$ for $t = 1, \ldots, T$ to be

$$v_t = \mathbf{1}^T x_t.$$

a) Formulate the problem as a linear dynamical system of the form

$$x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T-1.$$

The control vector u_t should have dimension n.

b) Assume that our trades incur quadratic transaction costs with parameter $\rho > 0$. For example, if at time t we buy $c > 0$ dollars of the first asset, and we sell $d > 0$ dollars of the second asset (the example above), then the transaction costs for the transition x_t to x_{t+1} are

$$\rho(c^2 + d^2).$$

(Be careful, they are **not** $\rho(c+d)^2$.) Explain how to solve the problem of maximizing the final value of the portfolio v_T minus the total transaction costs. (The sequence of controls $u_1, \ldots, u_{T-1}$ that achieves the maximum should be a function of A, B, and ρ). Use methods from EE263.

c) Apply your method to the following data.

```
T = 12;
x_1 = [1000, 1000, 0, 1000, 0, 0];
mu = [1.001, 1.003, 1.004, 1.006, 1.007];
rho = 0.0001;
```

What is the final value v_T? What are the total transaction costs? Plot the trajectories of the portfolio positions x_t. (On the same plot you should draw $n+1$ lines, one for each of the assets and cash, with time on the x-axis.)

d) Now assume that we aim to *liquidate* an initial portfolio, which means that at time T we want to have zero positions in any of the n assets and only hold cash. We thus impose the constraint $(x_T)_i = 0$, for $i = 1, \ldots, n$. Explain how to solve the problem of maximizing the final portfolio value (in this case, all cash) minus the transaction costs with this additional constraint. Use methods from EE263.

e) Apply your method to the data given above. What is the final value v_T? What are the total transaction costs? Plot the trajectories of the portfolio positions x_t. (On the same plot you should draw $n+1$ lines, one for each of the assets and cash, with time on the x-axis.)

Solution.

a) We define the vector

$$\tilde{\mu} = \begin{bmatrix} \mu \\ 1 \end{bmatrix},$$

then

$$\tilde{x}_t = \mathrm{diag}(\tilde{\mu}) x_t,$$

and we call

$$A = \mathrm{diag}(\tilde{\mu}).$$

We represent the trades by a vector $u_t \in \mathbb{R}^n$ for $t = 1, \ldots, T-1$. Each element is the dollar amount of each of the assets that we exchange for cash. Then the matrix B is

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -1 & -1 & \cdots & -1 \end{bmatrix}.$$

We thus have that

$$x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T-1.$$

b) The final state is given by

$$x_T = A^{T-1}x_1 + \sum_{t=1}^{T-1} A^{T-t-1}Bu_t$$

or equivalently

$$x_T = A^{T-1}x_1 + \mathcal{B}^T\mathcal{U}$$

with

$$\mathcal{B}^T = \left[A^{T-2}B \; A^{T-3}B \; \cdots \; B\right], \quad \mathcal{U} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{T-1} \end{bmatrix}.$$

Thus, the final value of the portfolio is given by

$$v_T = \mathbf{1}^T A^{T-1}x_1 + \mathbf{1}^T\mathcal{B}^T\mathcal{U}.$$

The quadratic transaction costs are

$$\rho\|\mathcal{U}\|^2.$$

The maximization problem is

$$\text{maximize} \quad v_T - \rho\|\mathcal{U}\|^2$$

since the first term of v_T is a constant, this is equivalent to

$$\text{maximize} \quad \mathbf{1}^T\mathcal{B}^T\mathcal{U} - \rho\|\mathcal{U}\|^2$$

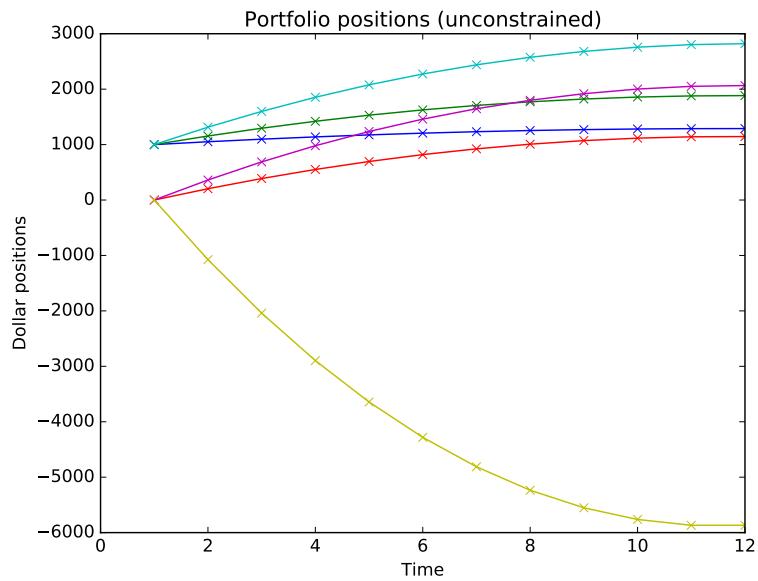
the solution is (by the first order condition, since the function is strictly concave)

$$\mathcal{U} = \frac{1}{2\rho}\mathcal{B}\mathbf{1}.$$

c) The final value and total transaction costs are

$$v_T = 3335.06, \quad \rho\|\mathcal{U}\|^2 = 111.24, \quad v_T - \rho\|\mathcal{U}\|^2 = 3223.81.$$

The yellow line is cash, the others are the asset positions.



d) The optimization problem is

$$\begin{array}{ll} \text{maximize} & v_T - \rho\|\mathcal{U}\|^2 \\ \text{s.t.} & (x_T)_i = 0 \quad i = 1, \ldots, n. \end{array}$$

Let $\bar{x}_t \in \mathbb{R}^n$ be the first n elements of each vector x_t. Then we have

$$\bar{x}_{t+1} = \operatorname{diag}(\mu)\bar{x}_t + Iu_t,$$

$$\bar{x}_T = \operatorname{diag}(\mu)^{T-1}\bar{x}_1 + \sum_{t=1}^{T-1} \operatorname{diag}(\mu)^{T-t-1}u_t$$

or

$$\bar{x}_T = \operatorname{diag}(\mu)^{T-1}\bar{x}_1 + \mathcal{C}^T\mathcal{U}$$

with

$$\mathcal{C} = \begin{bmatrix} \operatorname{diag}(\mu)^{T-2} \\ \operatorname{diag}(\mu)^{T-3} \\ \vdots \\ I \end{bmatrix}.$$

Thus the optimization problem is

$$\begin{array}{ll} \text{maximize} & v_T - \rho\|\mathcal{U}\|^2 \\ \text{s.t.} & \mathcal{C}^T\mathcal{U} = -\operatorname{diag}(\mu)^{T-1}\bar{x}_1. \end{array}$$

We could solve this problem by introducing a Lagrange multiplier, but instead we transform the problem by subtracting the optimal solution to (b)

$$\mathcal{U}' = \mathcal{U} - \frac{1}{2\rho}\mathcal{B}\mathbf{1}$$

The objective function is

$$\mathbf{1}^T\mathcal{B}^T(\mathcal{U}' + \frac{1}{2\rho}\mathcal{B}\mathbf{1}) - \rho\|\mathcal{U}' + \frac{1}{2\rho}\mathcal{B}\mathbf{1}\|^2.$$

Expanding the squared norm and ignoring constant terms we obtain that the problem is equivalent to

$$\begin{array}{ll} \text{maximize} & -\rho\|\mathcal{U}'\|^2 \\ \text{s.t.} & \mathcal{C}^T\mathcal{U}' = y \end{array}$$

where

$$y = -\operatorname{diag}(\mu)^{T-1}\bar{x}_1 - \frac{1}{2\rho}\mathcal{C}^T\mathcal{B}\mathbf{1}.$$

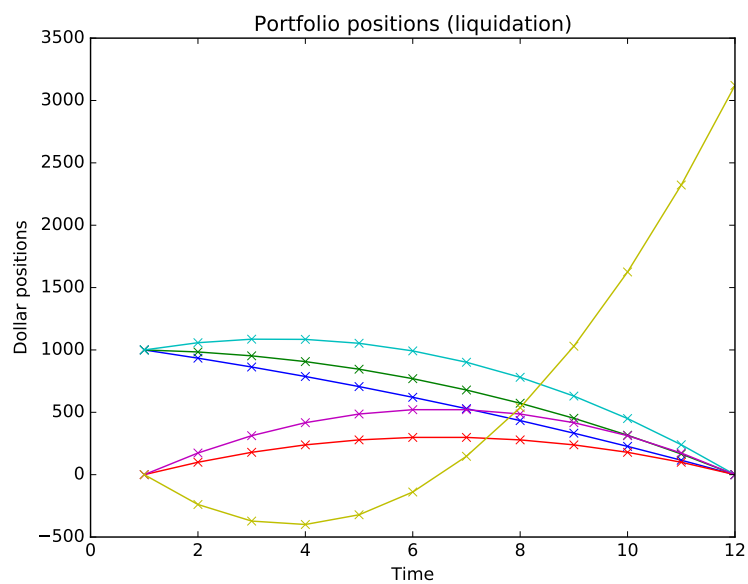
Note that $\mathcal{C}$ is full rank. The solution (least-norm underdetermined system) is

$$\mathcal{U}' = \mathcal{C}(\mathcal{C}^T\mathcal{C})^{-1}y$$

and

$$\mathcal{U} = \mathcal{C}(\mathcal{C}^T\mathcal{C})^{-1}y + \frac{1}{2\rho}\mathcal{B}\mathbf{1}.$$

e) The final value and total transaction costs are

$$v_T = 3120.65, \quad \rho\|\mathcal{U}\|^2 = 58.72, \quad v_T - \rho\|\mathcal{U}\|^2 = 3061.94.$$

The yellow line is cash, the others are the asset positions.