EE263 Homework 3

Summer 2024

3.300. Orthogonal complement of a subspace. If $\mathcal{V}$ is a subspace of $\mathbb{R}^n$ we define $\mathcal{V}^\perp$ as the set of vectors orthogonal to every element in $\mathcal{V}$, *i.e.*,

$$\mathcal{V}^\perp = \{\ x \mid \langle x, y\rangle = 0,\ \forall y \in \mathcal{V}\ \}.$$

a) Verify that $\mathcal{V}^\perp$ is a subspace of $\mathbb{R}^n$.

b) Suppose $\mathcal{V}$ is described as the span of some vectors $v_1, v_2, \ldots, v_r$. Express $\mathcal{V}$ and $\mathcal{V}^\perp$ in terms of the matrix $V = \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} \in \mathbb{R}^{n\times r}$ using common terms (range, nullspace, transpose, etc.)

c) Show that every $x \in \mathbb{R}^n$ can be expressed uniquely as $x = v + v^\perp$ where $v \in \mathcal{V}$, $v^\perp \in \mathcal{V}^\perp$. *Hint:* let v be the projection of x on $\mathcal{V}$.

d) Show that $\dim \mathcal{V}^\perp + \dim \mathcal{V} = n$.

e) Show that $\mathcal{V} \subseteq \mathcal{U}$ implies $\mathcal{U}^\perp \subseteq \mathcal{V}^\perp$.

Solution.

a) We do not need to check all properties of a vector space to hold for $\mathcal{V}^\perp$, since many of them hold only because $\mathcal{V}^\perp \subseteq \mathbb{R}^n$ and the vector sum and scalar product definitions over $\mathcal{V}^\perp$ and $\mathbb{R}^n$ are the same. We only need to verify the following properties:

- $0 \in \mathcal{V}^\perp$.
- $\forall x_1, x_2 \in \mathcal{V}^\perp :\ x_1 + x_2 \in \mathcal{V}^\perp$.
- $\forall \alpha \in \mathbb{R},\ \forall x \in \mathcal{V}^\perp :\ \alpha x \in \mathcal{V}^\perp$.

The first property comes from the fact that $\langle 0, y\rangle = 0$ for all $y \in \mathbb{R}^n$ and therefore $0 \in \mathcal{V}^\perp$. To verify the second property, we pick two arbitrary elements x_1 and x_2 in $\mathcal{V}^\perp$ and show that $x_1 + x_2 \in \mathcal{V}^\perp$. Let y be any vector in $\mathbb{R}^n$. We have

$$\begin{aligned}
\langle x_1 + x_2, y\rangle &= \langle x_1, y\rangle + \langle x_2, y\rangle \\
&= 0 + 0 \qquad\qquad (\text{since } x_1 \in \mathcal{V}^\perp \text{ and } x_2 \in \mathcal{V}^\perp) \\
&= 0,
\end{aligned}$$

and therefore $x_1 + x_2 \in \mathcal{V}^\perp$. Finally, we show that if $\alpha \in \mathbb{R}$ and $x \in \mathcal{V}^\perp$ then $\alpha x \in \mathcal{V}^\perp$. If $y \in \mathbb{R}^n$ is arbitrary

$$\begin{aligned}
\langle \alpha x, y\rangle &= \alpha\langle x, y\rangle \\
&= \alpha \cdot 0 \qquad\qquad (\text{since } x \in \mathcal{V}^\perp) \\
&= 0,
\end{aligned}$$

which by definition of $\mathcal{V}^\perp$, proves that $\alpha x \in \mathcal{V}^\perp$ and we are done.

b) Expressing $\mathcal{V}$ in terms of the matrix V is easy. The span of vectors $v_1, v_2, \ldots, v_r$ is simply all linear combinations of the columns of V and therefore $\mathcal{V} = \text{range}(V)$. To express $\mathcal{V}^\perp$ in terms of V we use the trivial fact that $x \in \mathcal{V}^\perp$ if and only if $x \perp v_i$ for $i = 1, \ldots, r$. (If $x \perp v_i$ then x is orthogonal to any linear combination of the v_i's and hence any element in $\mathcal{V}^\perp$. If $x \in \mathcal{V}^\perp$ then x is specially orthogonal to the vectors $v_i \in \mathcal{V}^\perp$ for $i = 1, \ldots, r$.) Therefore $x \in \mathcal{V}^\perp$ if and only if $v_i^\mathsf{T} x = 0$ for $i = 1, \ldots, r$. In other words, using matrix notation, $x \in \mathcal{V}^\perp$ if and only if

$$\begin{bmatrix} v_1^\mathsf{T} \\ v_2^\mathsf{T} \\ \vdots \\ v_r^\mathsf{T} \end{bmatrix} x = 0$$

or $V^\mathsf{T} x = 0$. Therefore $\mathcal{V}^\perp = \text{null}(V^\mathsf{T})$.

c) Suppose that $w_1, w_2, \ldots, w_k$ is an orthonormal basis for $\mathcal{V}$. Consider the projection of x on $\mathcal{V}$, *i.e.*,

$$v := (w_1^\mathsf{T} x) w_1 + (w_2^\mathsf{T} x) w_2 + \cdots + (w_k^\mathsf{T} x) w_k.$$

Clearly, $v \in \mathcal{V}$ because it is a linear combination of the basis vectors w_i. Now we show that $x - v$ (projection error) is an element in $\mathcal{V}^\perp$. To do this we only have to verify that $x - v \perp w_i$ or $w_i^\mathsf{T}(x - v) = 0$ for $i = 1, \ldots, k$. This is easy because

$$\begin{aligned} w_i^\mathsf{T}(x - v) &= w_i^\mathsf{T} x - w_i^\mathsf{T} v \\ &= w_i^\mathsf{T} x - (w_i^\mathsf{T} x) w_i^\mathsf{T} w_i && \text{since } w_i^\mathsf{T} w_j = 0 \text{ for } i \neq j \\ &= 0 && \text{since } w_i^\mathsf{T} w_i = 1 \end{aligned}$$

Now that $x - v \in \mathcal{V}^\perp$, define $v^\perp \in \mathcal{V}^\perp$ as $v^\perp = x - v$ so $x = v + v^\perp$ with $v \in \mathcal{V}$ and $v^\perp \in \mathcal{V}^\perp$. Now we show that the decomposition $x = v + v^\perp$ is unique. Suppose that there are two ways to express x as the sum of elements in $\mathcal{V}$ and $\mathcal{V}^\perp$, *i.e.*, $x = v_1 + v_1^\perp$ and $x = v_2 + v_2^\perp$ where $v_1, v_2 \in \mathcal{V}$ and $v_1^\perp, v_2^\perp \in \mathcal{V}^\perp$. Therefore $v_1 + v_1^\perp = v_2 + v_2^\perp$ or $v_1 - v_2 = v_1^\perp - v_2^\perp$. But $v_1 - v_2 \in \mathcal{V}$ (because $v_1, v_2 \in \mathcal{V}$) and $v_1^\perp - v_2^\perp \in \mathcal{V}^\perp$ (because $v_1^\perp, v_2^\perp \in \mathcal{V}^\perp$), and by definition of $\mathcal{V}^\perp$ we should have $(v_1 - v_2) \perp (v_1^\perp - v_2^\perp)$ or $(v_1 - v_2)^\mathsf{T}(v_1^\perp - v_2^\perp) = 0$. Now since $v_1 - v_2 = v_1^\perp - v_2^\perp$ this implies that

$$(v_1 - v_2)^\mathsf{T}(v_1 - v_2) = \|v_1 - v_2\|^2 = 0$$

and

$$(v_1^\perp - v_2^\perp)^\mathsf{T}(v_1^\perp - v_2^\perp) = \|v_1^\perp - v_2^\perp\|^2 = 0$$

so $v_1 = v_2$ and $v_1^\perp = v_2^\perp$ or the decomposition is *unique*.

d) This follows from the previous part. In part (**??**) we showed that any vector in $\mathbb{R}^n$ can be expressed as the sum of two elements in $\mathcal{V}$ and $\mathcal{V}^\perp$. Therefore, if $\{w_i\}_{i=1}^k$ is a basis for $\mathcal{V}$ and $\{u_i\}_{i=1}^l$ is a basis for $\mathcal{V}^\perp$, for arbitrary $x \in \mathbb{R}^n$ the scalars α_i and β_i exist such that

$$x = \sum_{i=1}^k \alpha_i w_i + \sum_{i=1}^l \beta_i u_i$$

or the set of vectors $\{w_1, \ldots, w_k, u_1, \ldots, u_l\}$ span $\mathbb{R}^n$. In fact, the vectors w_i for $i = 1, \ldots, k$ are orthogonal to the vectors u_i for $i = 1, \ldots, l$ by the definition of $\mathcal{V}^\perp$ and are therefore independent. Since the set of vectors $\{w_1, \ldots, w_k, u_1, \ldots, u_l\}$ span $\mathbb{R}^n$ and $w_1, \ldots, w_k, u_1, \ldots, u_l$ are independent we get

$$\dim \mathcal{V} + \dim \mathcal{V}^\perp = k + l = n.$$

e) To show that $\mathcal{U}^\perp \subseteq \mathcal{V}^\perp$ we take an arbitrary element $x \in \mathcal{U}^\perp$ and prove that $x \in \mathcal{V}^\perp$. Since $x \in \mathcal{U}^\perp$ then $x \perp y$ for all $y \in \mathcal{U}$. But $\mathcal{V} \subseteq \mathcal{U}$ so we also have $x \perp y$ for all $y \in \mathcal{V}$. By definition of $\mathcal{V}^\perp$, this is nothing but to state that $x \in \mathcal{V}^\perp$ and we are done.

3.350. Right inverses. This problem concerns the specific matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

This matrix is full rank (*i.e.*, its rank is 3), so there exists at least one right inverse. In fact, there are many right inverses of A, which opens the possibility that we can seek right inverses that in addition have other properties. For each of the cases below, either find a specific matrix $B \in \mathbb{R}^{5\times 3}$ that satisfies $AB = I$ and the given property, or explain why there is no such B. In cases where there is a right inverse B with the required property, you must briefly explain how you found your B. You must also attach a printout of your Julia script or Jupyter notebook showing the verification that $AB = I$. (We'll be very angry if we have to type in your 5×3 matrix into Julia to check it.) When there is no right inverse with the given property, briefly explain why there is no such B.

a) The second row of B is zero.

b) The nullspace of B has dimension one.

c) The third column of B is zero.

d) The second and third rows of B are the same.

e) B is upper triangular, *i.e.*, $B_{ij} = 0$ for $i > j$.

f) B is lower triangular, *i.e.*, $B_{ij} = 0$ for $i < j$.

Solution.

a) The second row of B is zero. This means that the second column of A isn't used in forming AB. Let $\tilde{A}$ be the matrix A with its second column removed, and let $\tilde{B}$ denote the matrix B with its second row (which is supposed to be zero) removed. We have $\tilde{A}\tilde{B} = AB = I$, so $\tilde{B}$ is a right inverse of $\tilde{A}$. There is such a matrix if and only if $\tilde{A}$ is

full rank, which it is. We can take $\tilde{B} = \tilde{A}^\mathsf{T}(\tilde{A}\tilde{A}^\mathsf{T})^{-1}$. Finally to construct B we simply insert a zero second row, moving rows 2, 3, 4 down by one. This gives the matrix

$$B = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}.$$

There are other possible choices as well.

b) The nullspace of B has dimension one. This means that B has rank 2, so the rank of AB is at most 2, which rules out the possibility that $AB = I$. So this is impossible.

c) The third column of B is zero. This implies B has a nullspace with dimension at least one, so by part (b) above, this is impossible too.

d) The second and third rows of B are the same. Let $\tilde{B}$ denote B with one of the (identical) rows 2 and 3 deleted. Then we have $AB = \tilde{A}\tilde{B}$, where $\tilde{A}$ is obtained from the matrix A by replacing its second column with the sum of its second and third columns, and deleting its third column. Thus, we need to find a right inverse for $\tilde{A}$, provided it is full rank. It is, so we can take $\tilde{B} = \tilde{A}^\mathsf{T}(\tilde{A}\tilde{A}^\mathsf{T})^{-1}$. Finally to construct B we simply insert a second copy of the second row of $\tilde{B}$ as a new third row. This gives

$$B = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}.$$

This matrix also happens to be the pseudo-inverse of A, $B = A^\mathsf{T}(AA^\mathsf{T})^{-1}$, and some of you noticed this immediately and used the pseudo-inverse to answer this question. That's a fine answer; it was our mistake to choose A so that the pseudo-inverse satisfied this condition. In general, of course, it would not.

e) B is upper triangular, *i.e.*, $B_{ij} = 0$ for $i > j$. If B is upper triangular, then it has the form

$$\begin{bmatrix} \tilde{B} \\ 0 \end{bmatrix},$$

where $\tilde{B}$ is square and upper triangular. If $AB = I$, then $\tilde{A}\tilde{B} = I$, where $\tilde{A}$ is the matrix formed from the first 3 columns of A. Thus we have $\tilde{A} = \tilde{B}^{-1}$. But the inverse of an upper triangular matrix is also upper triangular, so unless $\tilde{A}$ is upper triangular (and it isn't, in this case), we can't possibly have $\tilde{A}\tilde{B} = I$. So there is no such B in this case.

f) B is lower triangular, *i.e.*, $B_{ij} = 0$ for $i < j$. Let's label the columns of B as

$$b_1, \qquad b_2 = \begin{bmatrix} 0 \\ \tilde{b}_2 \end{bmatrix}, \qquad b_3 = \begin{bmatrix} 0 \\ 0 \\ \tilde{b}_3 \end{bmatrix},$$

where $\tilde{b}_2 \in \mathbb{R}^4$ and $\tilde{b}_3 \in \mathbb{R}^3$. To say that $AB = I$ is the same as saying that $Ab_1 = e_1$, $Ab_2 = e_2$, and $Ab_3 = e_3$, where e_1, e_2, e_3 are the unit vectors. We can solve these equations separately. The first equation is easy; the second we reduce to $\tilde{A}\tilde{b}_2 = e_2$, where here $\tilde{A}$ is A with its first column removed. The third is handled similarly. These equations do have a solution; we get

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Another way: we set it up as a set of 9 linear equations (one for each entry of $AB = I$) in $5 + 4 + 3 = 12$ variables. The variables are the first column of B (with 5 entries), the nonzero part of the second column of B (with 4 entries), and the nonzero part of the third second column of B (with 3 entries). We then attempt to solve these 9 equations in 12 variables. Some equations immediately give us the B matrix coefficients, while the others can be solved by inspection to obtain a rather simple matrix

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

3.520. Transmit powers in a wireless network. We consider a network of n transmitter/receiver pairs. Transmitter i transmits at power level p_i, which must satisfy $0 \le p_i \le P^{\max}$, where $P^{\max}$ is a given maximum transmitter power (which is the same for all transmitters). The path gain from transmitter j to receiver i is G_{ij} (which are all nonnegative, and G_{ii} are positive). The signal power at receiver i is given by $s_i = G_{ii}p_i$. The noise plus interference power at receiver i is given by

$$q_i = \sigma + \sum_{j \neq i} G_{ij} p_j$$

where $\sigma > 0$ is the self-noise power of the receivers (assumed to be the same for all receivers). The *signal to interference plus noise ratio* (SINR) at receiver i is defined as $S_i = s_i/q_i$.

a) Explain how to determine if there is a power allocation (*i.e.*, a vector p) that satisfies the constraints $0 \le p_i \le P^{\max}$ and achieves $S_i = S^{\text{target}}$ for $i = 1, \ldots, n$, where S^{target} is a (positive) target value of SINR. Explain how to find such a power allocation when it exists. You can assume that a matrix appearing in your analysis is full rank, but please make this assumption explicit.

b) Among the SINR target values $S^{\text{target}} = 2, 2.1, 2.2, \ldots, 3.9, 4$, find the largest for which there is a power allocation that satisfies the constraints $0 \le p_i \le P^{\max}$ and achieves $S_i = S^{\text{target}}$ for $i = 1, \ldots, n$, for the problem data

$$G = \begin{bmatrix} 1 & .2 & .1 \\ .1 & 2 & .1 \\ .3 & .1 & 3 \end{bmatrix}, \qquad \sigma = 0.01, \qquad P^{\max} = 0.1.$$

Remarks.

- Yes, this problem includes constraints on p (*i.e.*, that its entries are nonnegative and no more than $P^{\max}$), which we have not covered in EE263. Still, you can solve it (with material we *have* covered).
- If you solve this problem using methods that are more advanced or complicated than needed, we will deduct points.

Solution.

a) We want to find p such that $s_i/q_i = S^{\text{target}}$ for each i. We restate this as $s_i = S^{\text{target}} q_i$ for each i. This occurs if and only if

$$G_{ii}p_i = S^{\text{target}}\left(\sigma + \sum_{i \neq j} G_{ij}p_j\right)$$

which we write as

$$G_{ii}p_i - \sum_{i \neq j} S^{\text{target}} G_{ij}p_j = S^{\text{target}}\sigma.$$

This is a set of n linear equations in n variables or unknowns (*i.e.*, p). Defining $H_{ii} = G_{ii}$, and $H_{ij} = -S^{\text{target}}G_{ij}$ for $i \neq j$, and $g_i = S^{\text{target}}\sigma$ for $i = 1, \ldots, n$, we can write the equations as

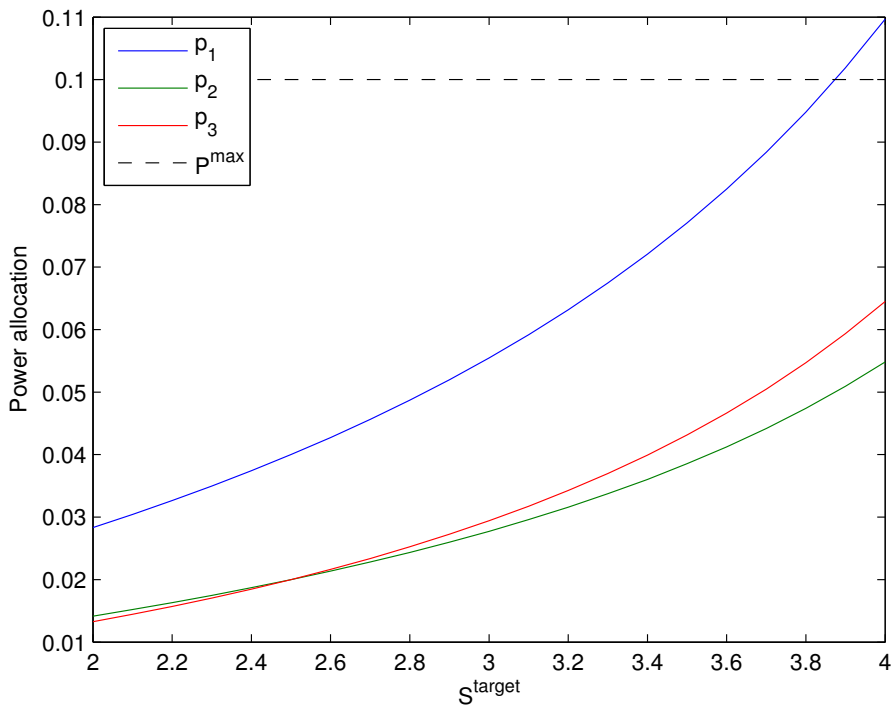
$$Hp = g.$$

We will assume that H is full rank (as the problem statement hints). In this case, there is only one value of p that satisfies these equations, given by $p = H^{-1}g$. If this p satisfies $0 \leq p_i \leq P^{\max}$, then p is a valid power allocation. If not, then no such p exists.

b) By plotting the powers versus S^{target}, or by direct observation, we find that the largest achievable SINR value from among those considered is $S^\star = 3.8$, with corresponding power allocation $p^\star = (0.0948, 0.0474, 0.0547)$. You can check that each H matrix is indeed invertible, so $Hp = g$ has a unique solution. The only thing we need to check is that the entries of p are within the bounds $0 \leq p_i \leq P^{\max}$.

The resulting plot is below. We see from the plot that, of the values considered, $S^{\text{target}} \leq 3.8$ has a feasible allocation, while values above 3.8 are infeasible, due to p_1 being too

large.



3.530. Checking some range and nullspace conditions. Explain how to determine whether or not the following statements hold:

a) $\text{range}(A) = \text{range}(B)$.

b) $\text{range}(A) \perp \text{range}(B)$.

c) $\text{range}(A) \cap \text{range}(B) = \{0\}$.

d) $\text{range}(C) \subseteq \text{null}(B)$.

The matrices have dimensions $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times p}$, $C \in \mathbb{R}^{p\times m}$.

Your answer can involve standard matrix operations on the matrices above, such as addition, multiplication, transposition, concatenation (*i.e.*, building block matrices), and inversion, as well as a function $\text{rank}(X)$, that gives the rank of a matrix X, and $\det(X)$, which gives the determinant of a (square) matrix X.

For example, you might assert that (a) holds if and only if $\text{rank}([A\ B]) = m$. (This is not correct; it's just an example of what your answer might look like.)

You do not need to give a proof or long justification that your conditions are correct; a short one or two sentence explanation for each statement is fine. Points will be deducted from correct answers that are substantially longer than they need to be, or are confusing (to us).

Solution.

a) $\text{range}(A) = \text{range}(B)$ if and only if $\text{rank}([A\ B]) = \text{rank}(A) = \text{rank}(B)$.

First assume that $\text{range}(A) = \text{range}(B)$ holds. In general we have $\text{range}([A\ B]) = \text{range}(A) + \text{range}(B)$. Since $\text{range}(A) = \text{range}(B)$, we have since $\text{range}([A\ B]) = \text{range}(A) = \text{range}(B)$ (this is equality of three subspaces of $\mathbb{R}^m$). Taking the dimension of these three (identical) subspaces, we conclude $\text{rank}([A\ B]) = \text{rank}(A) = \text{rank}(B)$.

To show the converse, suppose that $\text{rank}([A\ B]) = \text{rank}(A) = \text{rank}(B)$. We always have $\text{range}([A\ B]) \supseteq \text{range}(A)$ (and same for B). From the assumption, these two subspaces have the same dimension, and we conclude they are equal, *i.e.*, $\text{range}([A\ B]) = \text{range}(A)$. The same holds for B, and we conclude that $\text{range}(A) = \text{range}(B)$.

b) $\text{range}(A) \perp \text{range}(B)$ if and only if $A^\mathsf{T} B = 0$.

First note that $\text{range}(A) \perp \text{range}(B)$ means that for any u and any v, we have $(Au) \perp (Bv)$. This is equivalent to saying $(Au)^\mathsf{T}(Bv) = u^\mathsf{T}(A^\mathsf{T}B)v = 0$ for any u and v, which occurs if and only if $A^\mathsf{T}B = 0$. Recall that this last statement follows from the fact that if $u^\mathsf{T}Cv = 0$ for all u and v, then, in particular, $c_{ij} = e_i^\mathsf{T} C e_j = 0$ for all i, j.

c) $\text{range}(A) \cap \text{range}(B) = \{0\}$ if and only if $\text{rank}([A\ B]) = \text{rank}(A) + \text{rank}(B)$.

It will be easier to see the above equivalence if we have an orthonormal basis for $\text{range}(A)$ and for $\text{range}(B)$. Thus, let Q_A and Q_B be matrices (coming from, say, a QR factorization) whose columns form a basis for $\text{range}(A)$ and $\text{range}(B)$, respectively. Note that the ranges of A and Q_A are the same, and so are their ranks. The same is true for B and Q_B. Also, note that $\text{rank}([A\ B]) = \text{rank}([Q_A\ Q_B])$.

With this set up, we can proceed. Note that $\text{range}(Q_A) \cap \text{range}(Q_B) = \{0\}$ if and only if $Q_A x = Q_B y$ is only solved when $Q_A x = 0 = Q_B y$. Since the columns of Q_A and Q_B form independent sets, this happens if and only if $x = 0$ and $y = 0$. If we let $z = -y$, we can rewrite this statement as

$$\begin{bmatrix} Q_A & Q_B \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = 0$$

if and only if $x = 0$ and $z = 0$, which is the same as saying that the columns of $[Q_A\ Q_B]$ are linearly independent, or $[Q_A\ Q_B]$ has full rank. That is,

$$\text{rank}([Q_A\ Q_B]) = \text{rank}(Q_A) + \text{rank}(Q_B).$$

(Each of these matrices is full rank, so the rank is just the number of columns.)

Now, we can finally conclude that $\text{range}(A) \cap \text{range}(B) = \text{range}(Q_A) \cap \text{range}(Q_B) = \{0\}$ if and only if

$$\text{rank}([A\ B]) = \text{rank}([Q_A\ Q_B]) = \text{rank}(Q_A) + \text{rank}(Q_B) = \text{rank}(A) + \text{rank}(B).$$

d) $\text{range}(C) \subseteq \text{null}(B)$ if and only if $BC = 0$.

Stating $\text{range}(C) \subseteq \text{null}(B)$ is equivalent to stating that for any u, $Cu \in \text{null}(B)$. This, in turn, is equivalent to $B(Cu) = 0$ for any u, which occurs if and only if $BC = 0$.

3.680. Coin collector robot. Consider a robot with unit mass which can move in a frictionless two dimensional plane. The robot has a constant unit speed in the y direction (towards north), and it is designed such that we can only apply force in the x direction. We will apply a force at time t given by f_j for $2j-2 \leq t < 2j$ where $j = 1, \ldots, n$, so that the applied force is constant over time intervals of length 2. The robot is at the origin at time $t = 0$ with zero velocity in the x direction.

There are $2n$ coins in the plane and the goal is to design a sequence of input forces for the robot to collect the maximum possible number of coins. The robot is designed such that it can collect the ith coin only if it exactly passes through the location of the coin (x_i, y_i). In this problem, we assume that $y_i = i$.

a) Find the coordinates of the robot at time t, where t is a positive integer. Your answer should be a function of t and the vector of input forces $f \in \mathbf{R}^n$.

b) Given a sequence of k coins $(x_1, y_1), \ldots, (x_{2n}, y_{2n})$, describe a method to find whether the robot can collect them.

c) For the data provided in `robot_coin_collector.json`, show that the robot cannot collect all the coins.

d) Suppose that there is an arrangement of the coins such that it is not possible for the robot to collect all the coins. Suggest a way to check if the robot can collect all but one of the coins.

e) Run your method on data in `robot_coin_collector.json` and report which coin cannot be collected. Report the input that results in collecting $2n-1$ coins. Plot the location of the coins and the location of the robot at integer times.

Solution.

a) The second coordinate at time t is simply equal to t.

Consider $A \in \mathbb{R}^{2n \times n}$ such that

$$A_{ij} = \begin{cases} 1 & j = \lfloor \frac{i+1}{2} \rfloor \\ 0 & \text{Otherwise.} \end{cases}$$

Then we will have $Af = [f_1, f_1, f_2, \cdots, f_n]$. Similar to the mass/force example, the first coordinate at time t will be equal to $b_t^T Af$ where

$$b_t = [t - \frac{1}{2}, \cdots, \frac{1}{2}, 0, \cdots, 0]^T.$$

b) According to part a, the only possible time to collect the ith coin is at time $t = y_i = i$. Define l_i to be the first coordinate of the location of the robot at time $t = i$. From part a, we see that

$$l_i = b_t^T Af.$$

Let $B \in \mathbb{R}^{n \times n}$ be a matrix whose ith column is b_i and define $C = B^T A$. Then we will have $l = Cf$.

Hence, we see that the necessary and sufficient condition to collect all the coins is that $x \in \text{range}(C)$. This can be simply examined with `rank([C x])==rank(C)`.

c) The code to solve parts c,e can be find at the bottom.

d) In part b, we saw that $l = Cf$. We know that there exists a sequence of input forces f such that all but one of the $2n$ equations are satisfied, but we don't know which one.

Let $x^{(i)}$ be the location vector x with the ith entry removed. Likewise, let $C^{(i)}$ be the transition matrix with the ith row of C removed. If we can collect all coins but the ith one, then we will certainly have $x^{(i)} \in \text{range}(C^{(i)})$. We will loop over the coins and see whether it's possible to collect all coins but one.

e) The following code solves the problem:

```
using LinearAlgebra
using Plots

include("readclassjson.jl")
data = readclassjson("robot_coin_collector.json")

n = data["n"]
x = data["x"]

BT = zeros(2*n, 2*n)

for i = 1:2*n
    BT[i,1:i] = i-1/2:-1:1/2
end

A = zeros(2*n,n)
for i=1:n
    A[2*i-1, i], A[2*i, i] = 1, 1
end
C = BT*A

if rank([C x])==rank(C)
    println("All coins can be collected!")
else
    println("All coins cannot be collected!")
end

for i=1:2*n
    xt = x[1:end .!= i]
    Ct = C[1:end .!= i,:]
    if rank([Ct xt])==rank(Ct)
        println("The robot can collect all coins but ", i, "th, ")
        print("and the input will be: \n")
        global input = round.(Ct\xt, digits=4)
        display(input)
```

```
    end
end

plot(C*input, 1:2*n, color = "red", label = "Robot Trajectory")
scatter!(x, 1:2*n, color = "blue", label = "Coin Locations")
```

We see that all coins but the 7th can be collected and the associated input will be

$$f = [1.0000, -4.0000, 7.0000, -10.0000, 20.0000, -35.0000].$$

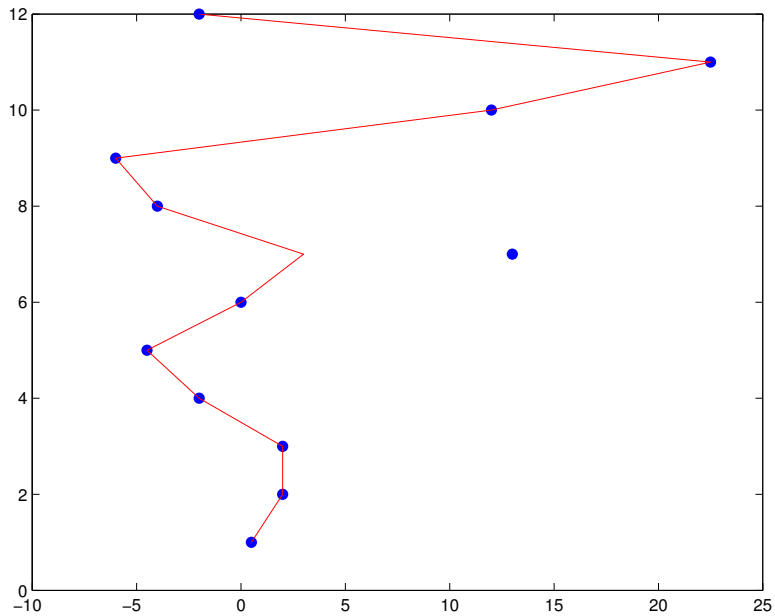


Figure 1: Location of the coins and the trajectory of the robot