EE263 Homework 2

Fall 2025

2.160. Some matrices from signal processing. We consider $x \in \mathbb{R}^n$ as a signal, with x_i the (scalar) value of the signal at (discrete) time period i, for $i = 1, \dots, n$. Below we describe several transformations of the signal x, that produce a new signal y (whose dimension varies). For each one, find a matrix A for which $y = Ax$.

a) $2\times$ *up-conversion with linear interpolation.* We take $y \in \mathbb{R}^{2n-1}$. For i odd, $y_i = x_{(i+1)/2}$. For i even, $y_i = (x_{i/2} + x_{i/2+1})/2$. Roughly speaking, this operation doubles the sample rate, inserting new samples in between the original ones using linear interpolation.

b) $2\times$ *down-sampling.* We assume here that n is even, and take $y \in \mathbb{R}^{n/2}$, with $y_i = x_{2i}$.

c) $2\times$ *down-sampling with averaging.* We assume here that n is even, and take $y \in \mathbb{R}^{n/2}$, with $y_i = (x_{2i-1} + x_{2i})/2$.

Solution.

a)

$$A_{\text{lin-int}} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

b)

$$A_{\text{down-sam}} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

c)

$$A_{\text{down-sam-avg}} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & \cdots & 0 & 0 \\ \vdots & & & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1/2 & 1/2 \end{bmatrix}$$

3.250. Color perception. Human color perception is based on the responses of three different types of color light receptors, called *cones*. The three types of cones have different spectral-response characteristics, and are called L, M, and, S because they respond mainly to long, medium, and short wavelengths, respectively. In this problem we will divide the visible spectrum into 20 bands, and model the cones' responses as follows:

$$L_{\text{cone}} = \sum_{i=1}^{20} l_i p_i, \qquad M_{\text{cone}} = \sum_{i=1}^{20} m_i p_i, \qquad S_{\text{cone}} = \sum_{i=1}^{20} s_i p_i,$$

where p_i is the incident power in the ith wavelength band, and l_i, m_i and s_i are nonnegative constants that describe the spectral responses of the different cones. The perceived color is a complex function of the three cone responses, *i.e.*, the vector $(L_{\text{cone}}, M_{\text{cone}}, S_{\text{cone}})$, with different cone response vectors perceived as different colors. (Actual color perception is a bit more complicated than this, but the basic idea is right.)

a) *Metamers.* When are two light spectra, p and $\tilde{p}$, visually indistinguishable? (Visually identical lights with different spectral power compositions are called *metamers*.)

b) *Visual color matching.* In a color matching problem, an observer is shown a test light, and is asked to change the intensities of three primary lights until the sum of the primary lights looks like the test light. In other words, the observer is asked the find a spectrum of the form

$$p_{\text{match}} = a_1 u + a_2 v + a_3 w,$$

where u, v, w are the spectra of the primary lights, and a_i are the intensities to be found, that is visually indistinguishable from a given test light spectrum p_{test}. Can this always be done? Discuss briefly.

c) *Visual matching with phosphors.* A computer monitor has three phosphors, R, G, and B. It is desired to adjust the phosphor intensities to create a color that looks like a reference test light. Find weights that achieve the match or explain why no such weights exist. The data for this problem is in `color_perception_data.json`, which contains the vectors `wavelength`, `B_phosphor`, `G_phosphor`, `R_phosphor`, `L_coefficients`, `M_coefficients`, `S_coefficients`, and `test_light`.

d) *Effects of illumination.* An object's surface can be characterized by its reflectance (*i.e.*, the fraction of light it reflects) for each band of wavelengths. If the object is illuminated with a light spectrum characterized by I_i, and the reflectance of the object is r_i (which is between 0 and 1), then the reflected light spectrum is given by $I_i r_i$, where $i = 1, \ldots, 20$ denotes the wavelength band. Now consider two objects illuminated (at different times) by two different light sources, say an incandescent bulb and sunlight. Sally argues that if the two objects look identical when illuminated by a tungsten bulb, then they will look identical when illuminated by sunlight. Beth disagrees: she says that two objects can appear identical when illuminated by a tungsten bulb, but look different when lit by sunlight. Who is right? If Sally is right, explain why. If Beth is right give an example of two objects that appear identical under one light source and different under another. You can use the vectors `sunlight` and `tungsten` defined in the data file as the light sources.

Remark. Spectra, intensities, and reflectances are all nonnegative quantities, which the material of EE263 doesn't address. So just ignore this while doing this problem. These issues can be handled using the material of EE364a, however.

Solution.

a) Let

$$A = \begin{bmatrix} l_1 & l_2 & l_3 & \cdots & l_{20} \\ m_1 & m_2 & m_3 & \cdots & m_{20} \\ s_1 & s_2 & s_3 & \cdots & s_{20} \end{bmatrix}.$$

Now suppose that $c = Ap$ is the cone response to the spectrum p and $\tilde{c} = A\tilde{p}$ is the cone response to spectrum $\tilde{p}$. If the spectra are indistinguishable, then $c = \tilde{c}$ and $Ap = A\tilde{p}$. Solving the last expression for zero gives $A(p - \tilde{p}) = 0$. In other words, p and $\tilde{p}$ are metamers if $(p - \tilde{p}) \in \text{null}(A)$.

b) In symbols, the problem asks if it is always possible to find nonnegative a_1, a_2, and a_3 such that

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = Ap_{\text{test}} = A \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Let $P = \begin{bmatrix} u & v & w \end{bmatrix}$ and let $B = AP$. If B is invertible, then

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = B^{-1} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}.$$

However, B is not necessarily invertible. For example, if $\text{rank}(A) < 3$ or $\text{rank}(P) < 3$ then B will be singular. Physically, A is full rank if the L, M, and S cone responses are linearly independent, which they are. The matrix P is full rank if and only if the spectra of the primary lights are independent. Even if both A and P are full rank, B could still be singular. Primary lights that generate an invertible B are called *visually independent*. If B is invertible, a_1, a_2, and a_3 exist that satisfy

$$Ap_{\text{test}} = A \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

One or more of the a_i may be negative in which case in the experimental setup described, no match would be possible. However, in a more complicated experimental setup that allows the primary lights to be combined either with each other or with p_{test}, a match is always possible if B is invertible. In this case, if $a_i < 0$, the ith light should be mixed with p_{test} instead of the other primary lights. For example, suppose $a_1 < 0$, $a_2, a_3 \geq 0$ and $b_1 = -a_1$, then

$$A(b_1 u + p_{\text{test}}) = A(a_2 v + a_3 w),$$

and each spectrum has a nonnegative weight.

c) Weights can be found as described above. The R, G, and B phosphors should be weighted by 0.4226, 0.0987, and 0.5286 respectively.

d) Beth is right. Let r and $\tilde{r}$ be the reflectances of two objects and let p and $\tilde{p}$ be two spectra. Let A be defined as before. Then, the objects will look identical under p if

$$A \underbrace{\begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & r_{20} \end{bmatrix}}_{R} p = A \underbrace{\begin{bmatrix} \tilde{r}_1 & 0 & \cdots & 0 \\ 0 & \tilde{r}_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \tilde{r}_{20} \end{bmatrix}}_{\tilde{R}} p.$$

This is equivalent to saying $(R - \tilde{R})p \in \text{null}(A)$. The objects will look different under $\tilde{p}$ if, additionally, $AR\tilde{p} \neq A\tilde{R}\tilde{p}$ which means that $(R - \tilde{R})\tilde{p} \notin \text{null}(A)$. The following code shows how to find reflectances r_1 and r_2 for two objects such that the objects will have the same color under tungsten light and will have different colors under sunlight.

Here is Julia code solving this problem.

```
using LinearAlgebra
include("readclassjson.jl");
include("qr.jl")

# return any vector in nullspace of A
# assume A is fat and full rank
# use null(A) = range(A')^\perp
function get_vector_in_nullspace(A)
    m, n = size(A)
    Q, R = fullqr(A')
    q = Q[:,m+1]
    return q
end

function main()
    # read data
    data = readclassjson("color_perception_data.json");
    L_coefficients = data["L_coefficients"];
    M_coefficients = data["M_coefficients"];
    S_coefficients = data["S_coefficients"];
    R_phosphor = data["R_phosphor"];
    G_phosphor = data["G_phosphor"];
    B_phosphor = data["B_phosphor"];
    test_light = data["test_light"];
    tungsten = data["tungsten"];
    sunlight = data["sunlight"];

    # create the main matrix
    A = [L_coefficients'; M_coefficients'; S_coefficients'];
```

```
    # PART C
    P = [R_phosphor G_phosphor B_phosphor]
    B = A*P
    @show pt_c_weights = B\(A*test_light)


    # PART D
    D_tungsten = diagm(tungsten)
    D_sunlight = diagm(sunlight)
    q = get_vector_in_nullspace(A*D_tungsten)
    # fix reflectance r1
    r1 = [0, 0.2, 0.3, 0.7, 0.7, 0.8, 0.8, 0.2, 0.9, 0.8,
          0.2, 0.8, 0.9, 0.2, 0.8, 0.3, 0.8, 0.7, 0.2, 0.4]

    # and make r2 related
    r2 = r1 + q

    # response of eye in tungsten
    # same for both, since they differ by a vector in the nullspace
    @show pt_d_r1_tungsten_response = A*D_tungsten*r1
    @show pt_d_r2_tungsten_response = A*D_tungsten*r2

    # and response to sunlight
    # differs for r1 vs r2
    @show pt_d_r1_sunlight_response = A*D_sunlight*r1
    @show pt_d_r2_sunlight_response = A*D_sunlight*r2

    return
end
```

3.260. Halfspace. Suppose $a, b \in \mathbb{R}^n$ are two given points. Show that the set of points in $\mathbb{R}^n$ that are closer to a than b is a halfspace, *i.e.*:

$$\{x \mid \|x-a\| \le \|x-b\|\ \} = \{\ x \mid c^\mathsf{T} x \le d\}$$

for appropriate $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. Give c and d explicitly, and draw a picture showing a, b, c, and the halfspace.

Solution. It is easy to see geometrically what is going on: the hyperplane that goes right between a and b splits $\mathbb{R}^n$ into two parts; the points closer to a (than b) and the points closer to b (than a). More precisely, the hyperplane is normal to the line through a and b, and intersects that line at the midpoint between a and b. Now that we have the idea, let's try to derive it algebraically. Let x belong to the set of points in $\mathbb{R}^n$ that are closer to a than b. Therefore $\|x-a\| < \|x-b\|$ or $\|x-a\|^2 < \|x-b\|^2$ so

$$(x-a)^\mathsf{T}(x-a) < (x-b)^\mathsf{T}(x-b).$$

Expanding the inner products gives

$$x^\mathsf{T}x - x^\mathsf{T}a - a^\mathsf{T}x + a^\mathsf{T}a < x^\mathsf{T}x - x^\mathsf{T}b - b^\mathsf{T}x + b^\mathsf{T}b$$

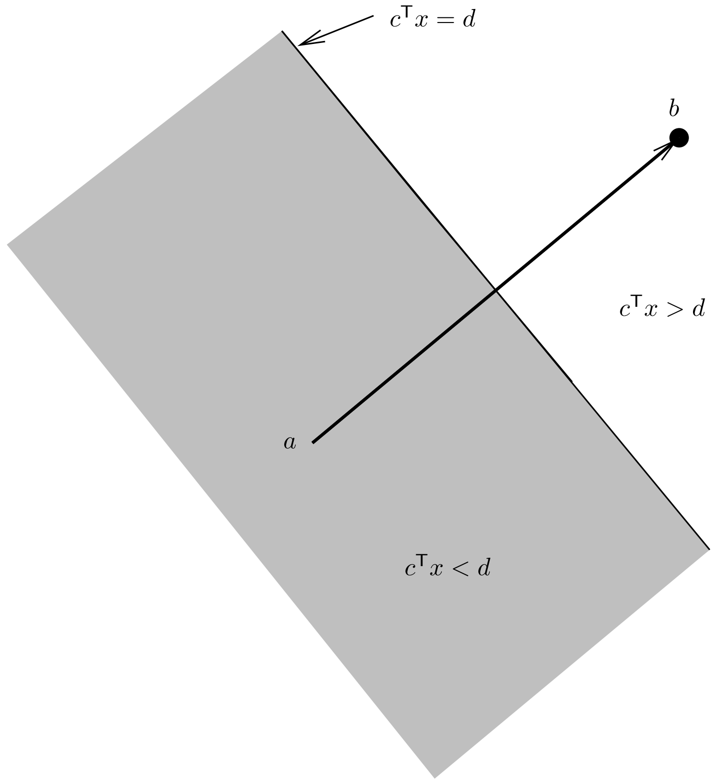
or

$$-2a^\mathsf{T}x + a^\mathsf{T}a < -2b^\mathsf{T}x + b^\mathsf{T}b$$

and finally

$$(b-a)^\mathsf{T}x < \frac{1}{2}(b^\mathsf{T}b - a^\mathsf{T}a). \tag{1}$$

Thus (1) is in the form $c^\mathsf{T}x < d$ with $c = b - a$ and $d = \frac{1}{2}(b^\mathsf{T}b - a^\mathsf{T}a)$ and therefore we have shown that the set of points in $\mathbb{R}^n$ that are closer to a than b is a halfspace. Note that the hyperplane $c^\mathsf{T} x = d$ is perpendicular to $c = b - a$.



3.300. Orthogonal complement of a subspace. If $\mathcal{V}$ is a subspace of $\mathbb{R}^n$ we define $\mathcal{V}^\perp$ as the set of vectors orthogonal to every element in $\mathcal{V}$, *i.e.*,

$$\mathcal{V}^\perp = \{ \ x \mid \langle x, y \rangle = 0, \ \forall y \in \mathcal{V} \ \}.$$

a) Verify that $\mathcal{V}^\perp$ is a subspace of $\mathbb{R}^n$.

b) Suppose $\mathcal{V}$ is described as the span of some vectors $v_1, v_2, \ldots, v_r$. Express $\mathcal{V}$ and $\mathcal{V}^\perp$ in terms of the matrix $V = \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} \in \mathbb{R}^{n \times r}$ using common terms (range, nullspace, transpose, etc.)

c) Show that every $x \in \mathbb{R}^n$ can be expressed uniquely as $x = v + v^\perp$ where $v \in \mathcal{V}$, $v^\perp \in \mathcal{V}^\perp$. *Hint:* let v be the projection of x on $\mathcal{V}$.

d) Show that $\dim \mathcal{V}^\perp + \dim \mathcal{V} = n$.

e) Show that $\mathcal{V} \subseteq \mathcal{U}$ implies $\mathcal{U}^\perp \subseteq \mathcal{V}^\perp$.

Solution.

a) To show that $\mathcal{V}^\perp$ is a subspace we only need to show that for all $p, q \in \mathcal{V}^\perp$ and all $\alpha \in \mathbb{R}$ we have

$$p + q \in \mathcal{V}^\perp \quad \text{and} \quad \alpha q \in \mathcal{V}^\perp$$

We have, for all $y \in \mathcal{V}$,

$$(p+q)^\mathsf{T} y = p^\mathsf{T} y + q^\mathsf{T} y = 0$$

which gives the first property. The second property holds because

$$(\alpha p)^\mathsf{T} y = \alpha p^\mathsf{T} y = 0$$

b) We have

$$\mathcal{V} = \text{range}(V)$$

since the subspace is the span of the columns. Also

$$\mathcal{V}^\perp = \text{range}(V)^\perp = \text{null}(V^\mathsf{T})$$

c) Let $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ be an orthogonal matrix with $\text{range}(Q_1) = \mathcal{V}$. Therefore $\mathcal{V}^\perp = \text{range}(Q_2)$. (Such a matrix can be constructed using the QR factorization of V.)

The statement

x can be expressed uniquely as $x = v + v^\perp$ where $v \in \mathcal{V}$, $v^\perp \in \mathcal{V}^\perp$

is equivalent to

there exists a unique y such that $x = Qy$

because $x = Qy$ means $x = Q_1 y_1 + Q_2 y_2$, and the columns of Q_1 are a basis for $\mathcal{V}$, and the columns of Q_2 are a basis for $\mathcal{V}^\perp$.

d) Using the matrix Q from the previous part, we have that the number of columns of Q_2 plus the number of columns of Q_1 is equal to n.

e) We would like to show that any $x \in \mathcal{U}^\perp$ also satisfies $x \in \mathcal{V}^\perp$. Assume $x \in \mathcal{U}^\perp$, then x is orthogonal to all vectors in $\mathcal{U}$, and, since $\mathcal{V} \subseteq \mathcal{U}$, we therefore have that x is orthogonal to all vectors in $\mathcal{V}$, and so $x \in \mathcal{V}^\perp$.

3.450. Minimum distance and maximum correlation decoding. We consider a simple communication system, in which a sender transmits one of N possible signals to a receiver, which receives a version of the signal sent that is corrupted by noise. Based on the corrupted received signal, the receiver has to estimate or guess which of the N signals was sent. We will represent the signals by vectors in $\mathbb{R}^n$. We will denote the possible signals as $a_1, \ldots, a_N \in \mathbb{R}^n$. These signals, which collectively are called the *signal constellation*, are known to both the transmitter and receiver. When the signal a_k is sent, the received signal is $a_{\text{recd}} = a_k + v$, where $v \in \mathbb{R}^n$ is (channel or transmission) noise. In a communications course, the noise v is described by a statistical model, but here we'll just assume that it is 'small' (and in any case, it does not matter for the problem). The receiver must make a guess or estimate as to which of the signals was sent, based on the received signal a_{recd}. There are many ways to do this, but in this problem we explore two methods.

- *Minimum distance decoding.* Choose as the estimate of the decoded signal the one in the constellation that is closest to what is received, *i.e.*, choose a_k that minimizes $\|a_{\text{recd}} - a_i\|$. For example, if we have $N = 3$ and

$$\|a_{\text{recd}} - a_1\| = 2.2, \qquad \|a_{\text{recd}} - a_2\| = 0.3, \qquad \|a_{\text{recd}} - a_3\| = 1.1,$$

then the minimum distance decoder would guess that the signal a_2 was sent.

- *Maximum correlation decoding.* Choose as the estimate of the decoded signal the one in the constellation that has the largest inner product with the received signal, *i.e.*, choose a_k that maximizes $a_{\text{recd}}^{\mathsf{T}} a_i$. For example, if we have $N = 3$ and

$$a_{\text{recd}}^{\mathsf{T}} a_1 = -1.1, \qquad a_{\text{recd}}^{\mathsf{T}} a_2 = 0.2, \qquad a_{\text{recd}}^{\mathsf{T}} a_3 = 1.0,$$

  then the maximum correlation decoder would guess that the signal a_3 was sent.

For both methods, let's not worry about breaking ties. You can just assume that ties never occur; one of the signals is always closest to, or has maximum inner product with, the received signal. Give some general conditions on the constellation (*i.e.*, the set of vectors $a_1, \ldots, a_N$) under which these two decoding methods are the same. By 'same' we mean this: for any received signal a_{recd}, the decoded signal for the two methods is the same. Give the simplest condition you can. You must show how the decoding schemes always give the same answer, when your conditions hold. Also, give a specific counterexample, for which your conditions don't hold, and the methods differ. (We are *not* asking you to show that when your conditions don't hold, the two decoding schemes differ for some received signal.) You might want to check simple cases like $n = 1$ (scalar signals), $N = 2$ (only two messages in the constellation), or draw some pictures. But then again, you might not.

Solution. The minimum distance decoder works like this: We choose our estimate, a_j, such that the j^{th} of the following N numbers is the minimum:

$$\|a_{\text{recd}} - a_j\|^2 = \|a_{\text{recd}}\|^2 - 2a_{\text{recd}}^{\mathsf{T}} a_j + \|a_j\|^2, \quad j = 1, \ldots, N.$$

That's the same as finding the smallest of the N numbers

$$-2a_{\text{recd}}^{\mathsf{T}} a_j + \|a_j\|^2, \quad j = 1, \ldots, N.$$

The maximum correlation decoder finds the maximum of the N numbers

$$a_{\text{recd}}^{\mathsf{T}} a_j, \quad j = 1, \ldots, N,$$

which gives the estimate. Now we can clearly see one simple condition under which the two decoders always yield the same result, independent of what $a_(recd)$ is: if $\|a_j\|$ is the *same* for all j, then the two methods above are the same. In other words: minimum distance decoding is the same as maximum correlation decoding if the signals in the constellation all have the same norm. It's easy to find a simple example where the two methods differ (and of course the condition we found above doesn't hold). We can take $a_1 = 1,\ a_2 = 2$. Then the minimum distance decoder selects $a_1 = 1$ whenever $a_{\text{recd}} < 1.5$ and selects $a_2 = 2$ whenever $a_{\text{recd}} > 1.5$ (remember that we don't worry about ties). The maximum correlation detector chooses $a_1 = 1$ whenever $a_{\text{recd}} < 0$ and chooses $a_2 = 2$ whenever $a_{\text{recd}} > 0$. For example, when $a_{\text{recd}} = 1$ the minimum distance decoder selects $a_1 = 1$; the maximum correlation detector selects $a_2 = 2$. Please note that we did not ask you which of the methods works better (which is something you'd study in a communications course); we only asked under what conditions are the methods the same. Any solution that involved discussion of a_{recd} or v, or presented conditions on them, couldn't satisfy our requirement that the methods be identical for all a_{recd}. By the way, the condition that the signals in the constellation all have the same norm is not only sufficient for the two methods to be the same; it is also necessary. But we didn't ask you to show that.