EE263 Homework 1

Fall 2025

2.61. Matrix representation of polynomial differentiation. We can represent a polynomial of degree less than n,

$$p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0,$$

as the vector $(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{R}^n$. Consider the linear transformation $\mathcal{D}$ that differentiates polynomials, *i.e.*, $\mathcal{D}p = dp/dx$. Find the matrix D that represents $\mathcal{D}$ (*i.e.*, if the coefficients of p are given by a, then the coefficients of dp/dx are given by Da).

Solution. We have

$$p'(x) = (n-1)a_{n-1}x^{n-2} + (n-2)a_{n-2}x^{n-3} + \cdots + a_1,$$

which corresponds to the vector $(a_1, 2a_2, \ldots, (n-1)a_{n-1}, 0)$ (noting that the x^{n-1} coefficient of p' is always zero). So D is the matrix that maps the vector $(a_0, a_1, \ldots, a_{n-1})$ into the vector $(a_1, 2a_2, \ldots, (n-1)a_{n-1}, 0)$. Equating coefficients, we have

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n-1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

2.100. A mass subject to applied forces. Consider a unit mass subject to a time-varying force $f(t)$ for $0 \le t \le n$. Let the initial position and velocity of the mass both be zero. Suppose that the force has the form $f(t) = x_j$ for $j-1 \le t < j$ and $j = 1, \ldots, n$. Let y_1 and y_2 denote, respectively, the position and velocity of the mass at time $t = n$.

a) Find the matrix $A \in \mathbb{R}^{2\times n}$ such that $y = Ax$.

b) For $n = 4$, find a sequence of input forces $x_1, \ldots, x_n$ that moves the mass to position 1 with velocity 0 at time n.

Solution. Let $p(t)$ and $v(t)$ denote, respectively, the position and velocity of the mass at time t.

a) The velocity is the integral of the applied force:

$$\begin{aligned} v(t) &= v(0) + \int_0^t f(\tau)\,d\tau \\ &= v(0) + \sum_{j=1}^{\lfloor t \rfloor} \int_{j-1}^{j} f(\tau)\,d\tau + \int_{\lfloor t \rfloor}^{t} f(\tau)\,d\tau \\ &= v(0) + \sum_{j=1}^{\lfloor t \rfloor} \int_{j-1}^{j} x_j\,d\tau + \int_{\lfloor t \rfloor}^{t} x_{\lfloor t \rfloor+1}\,d\tau \\ &= v(0) + \sum_{j=1}^{\lfloor t \rfloor} (\tau x_j)\big|_{\tau=j-1}^{\tau=j} + (\tau x_{\lfloor t \rfloor+1})\big|_{\tau=\lfloor t \rfloor}^{\tau=t} \\ &= v(0) + \sum_{j=1}^{\lfloor t \rfloor} x_j + (t - \lfloor t \rfloor) x_{\lfloor t \rfloor+1}. \end{aligned}$$

In particular, because the mass is initially at rest (that is, $v(0) = 0$), the final velocity is

$$y_2 = v(n) = \sum_{j=1}^{n} x_j.$$

Similarly, the position is the integral of the velocity:

$$\begin{aligned} p(t) &= p(0) + \int_0^t v(\tau)\,d\tau \\ &= p(0) + \int_0^t (v(0) + (v(\tau) - v(0)))\,d\tau \\ &= p(0) + v(0)t + \int_0^t (v(\tau) - v(0))\,d\tau \\ &= p(0) + v(0)t + \sum_{j=1}^{\lfloor \tau \rfloor} \int_{j-1}^{j} (v(\tau) - v(0))\,d\tau + \int_{\lfloor t \rfloor}^{t} (v(\tau) - v(0))\,d\tau \\ &= p(0) + v(0)t + \sum_{j=1}^{\lfloor t \rfloor} \int_{j-1}^{j} \left(\sum_{k=1}^{\lfloor \tau \rfloor} x_k + (\tau - \lfloor \tau \rfloor) x_{\lfloor \tau \rfloor+1} \right) d\tau \\ &\quad + \int_{\lfloor t \rfloor}^{t} \left(\sum_{k=1}^{\lfloor \tau \rfloor} x_k + (\tau - \lfloor \tau \rfloor) x_{\lfloor \tau \rfloor+1} \right) d\tau \\ &= p(0) + v(0)t + \sum_{j=1}^{\lfloor t \rfloor} \int_{j-1}^{j} \left(\sum_{k=1}^{j-1} x_k + (\tau - (j-1)) x_j \right) d\tau \end{aligned}$$

$$
\begin{aligned}
&+ \int_{\lfloor t \rfloor}^{t} \left(\sum_{k=1}^{\lfloor t \rfloor} x_k + (\tau - \lfloor t \rfloor) x_{\lfloor t \rfloor + 1} \right) d\tau \\
&= p(0) + v(0)t + \sum_{k=1}^{\lfloor t \rfloor} \left(\sum_{k=1}^{j-1} \tau x_k + \frac{1}{2}(\tau - (j-1))^2 x_j \right) \Bigg|_{\tau = j-1}^{\tau = j} \\
&+ \left(\sum_{k=1}^{\lfloor t \rfloor} \tau x_k + \frac{1}{2}(\tau - \lfloor t \rfloor)^2 x_{\lfloor t \rfloor + 1} \right) \Bigg|_{\tau = \lfloor t \rfloor}^{\tau = t} \\
&= p(0) + v(0)t + \sum_{j=1}^{\lfloor t \rfloor} \left(\sum_{k=1}^{j-1} x_k + \frac{1}{2} x_j \right) + \left(\sum_{k=1}^{\lfloor t \rfloor} x_k + \frac{1}{2}(t - \lfloor t \rfloor)^2 x_{\lfloor t \rfloor + 1} \right) \\
&= p(0) + v(0)t + \sum_{j=1}^{\lfloor t \rfloor} \sum_{k=1}^{j-1} x_k + \sum_{j=1}^{\lfloor t \rfloor} \frac{1}{2} x_j + \sum_{k=1}^{\lfloor t \rfloor} x_k + \frac{1}{2}(t - \lfloor t \rfloor)^2 x_{\lfloor t \rfloor + 1} \\
&= p(0) + v(0)t + \sum_{k=1}^{\lfloor t \rfloor} \sum_{j=k+1}^{\lfloor t \rfloor} x_k + \sum_{k=1}^{\lfloor t \rfloor} \frac{1}{2} x_k + \sum_{k=1}^{\lfloor t \rfloor} x_k + \frac{1}{2}(t - \lfloor t \rfloor)^2 x_{\lfloor t \rfloor + 1} \\
&= p(0) + v(0)t + \sum_{k=1}^{\lfloor t \rfloor} (\lfloor t \rfloor - k) x_k + \sum_{k=1}^{\lfloor t \rfloor} \frac{1}{2} x_k + \sum_{k=1}^{\lfloor t \rfloor} x_k + \frac{1}{2}(t - \lfloor t \rfloor)^2 x_{\lfloor t \rfloor + 1} \\
&= p(0) + v(0)t + \sum_{k=1}^{\lfloor t \rfloor} \left((\lfloor t \rfloor - k) + \frac{1}{2} + (t - \lfloor t \rfloor) \right) x_k + \frac{1}{2}(t - \lfloor t \rfloor)^2 x_{\lfloor t \rfloor + 1} \\
&= p(0) + v(0)t + \sum_{k=1}^{\lfloor t \rfloor} \left(t - k + \frac{1}{2} \right) x_k + \frac{1}{2}(t - \lfloor t \rfloor)^2 x_{\lfloor t \rfloor + 1}.
\end{aligned}
$$

In particular, because the mass is initially at rest at the origin (that is, $p(0) = 0$ and $v(0) = 0$), the final position is

$$
y_1 = p(n) = \sum_{j=1}^{n} \left(n - k + \tfrac{1}{2} \right) x_j.
$$

Thus, we obtain the following system of linear equations:

$$
\begin{aligned}
y_1 &= \sum_{j=1}^{n} (n - j + \tfrac{1}{2}) x_j, \\
y_2 &= \sum_{j=1}^{n} x_j.
\end{aligned}
$$

Since A_{ij} gives the coefficient of x_j in our expression for y_i, we have that

$$
A_{1j} = n - j + \tfrac{1}{2} \qquad \text{and} \qquad A_{2j} = 1, \qquad j = 1, \ldots, n.
$$

More concretely, we have that

$$A = \begin{bmatrix} n - \frac{1}{2} & n - \frac{3}{2} & \cdots & \frac{3}{2} & \frac{1}{2} \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

b) We want to solve the following system of linear equations:

$$\begin{bmatrix} \frac{7}{2} & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

This system is underdetermined, and has infinitely many solutions. Suppose we choose $x_2 = x_3 = 0$. Then, we are left with the system

$$\begin{bmatrix} \frac{7}{2} & \frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The second equation implies that $x_4 = -x_1$. Then, the first equation becomes

$$\tfrac{7}{2}x_1 + \tfrac{1}{2}x_4 = \tfrac{7}{2}x_1 - \tfrac{1}{2}x_1 = 3x_1 = 1.$$

Solving this equation, we find that $x_1 = \frac{1}{3}$. Substituting this value into our expression for x_4 gives $x_4 = -x_1 = -\frac{1}{3}$. Thus, one sequence of input forces that moves the mass to position 1 with velocity 0 at time n is

$$x = \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \\ -\frac{1}{3} \end{bmatrix}.$$

2.110. Counting paths in an undirected graph. Consider an undirected graph with n nodes, and no self loops (*i.e.*, all branches connect two different nodes). Let $A \in \mathbf{R}^{n \times n}$ be the *node adjacency matrix*, defined as

$$A_{ij} = \begin{cases} 1 & \text{if there is a branch from node } i \text{ to node } j \\ 0 & \text{if there is no branch from node } i \text{ to node } j \end{cases}$$

Note that $A = A^{\mathsf{T}}$, and $A_{ii} = 0$ since there are no self loops. We can intrepret A_{ij} (which is either zero or one) as the number of branches that connect node i to node j. Let $B = A^k$, where $k \in \mathbb{Z}$, $k \geq 1$. Give a simple interpretation of B_{ij} in terms of the original graph. (You might need to use the concept of a *path* of length m from node p to node q.)

Solution. First consider $B = A$, *i.e.*, ($k = 1$). Obviously, the interpretation of B_{ij} is the number of branches that connect node i to node j (either 0 or 1). In other words, B_{ij} is equal to the number of paths of length 1 that connect node i to node j. Now consider the case $k = 2$ so $B = A^2$ and

$$B_{ij} = \sum_m A_{im} A_{mj}.$$

Clearly, B_{ij} becomes the number of paths of length 2 from node i to node j. $A_{im}A_{mj}$ is nonzero only when both A_{im} and A_{mj} are nonzero so that there exists a path of length 2 from node i to node j via node m. The summation is over *all* nodes m and $A_{im}A_{mj}$ is either 0 or 1, so in fact, B_{ij} sums up to the number of paths of length 2 from node i to node j. For $k = 3$, $B = A^3$ and therefore

$$B_{ij} = \sum_{m_1} \sum_{m_2} A_{im_1} A_{m_1 m_2} A_{m_2 j}.$$

For similar reasons, B_{ij} now becomes the number of paths of length 3 from node i to node j. The summation is over all intermediate nodes m_1 and m_2, and $A_{im_1} A_{m_1 m_2} A_{m_2 j} = 1$ means that there is a path of length 3 from node i to node j via nodes m_1 and m_2. In general, for $B = A^k$, B_{ij} has the interpretation of "the number of paths of length k from node i to j" because

$$B_{ij} = \sum_{m_1} \sum_{m_2} \cdots \sum_{m_{k-1}} A_{im_1} A_{m_1 m_2} \cdots A_{m_{k-1} j}$$

and $A_{im_1} A_{m_1 m_2} \cdots A_{m_{k-1} j} = 1$ means that there is a path of length k from node i to node j via nodes $m_1, m_2, \ldots, m_{k-1}$.

2.150. Gradient of some common functions. Recall that the gradient of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, at a point $x \in \mathbb{R}^n$, is defined as the vector

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix},$$

where the partial derivatives are evaluated at the point x. The first order Taylor approximation of f, near x, is given by

$$\hat{f}_{\text{tay}}(z) = f(x) + \nabla f(x)^{\mathsf{T}}(z - x).$$

This function is affine, *i.e.*, a linear function plus a constant. For z near x, the Taylor approximation $\hat{f}_{\text{tay}}$ is very near f. Find the gradient of the following functions. Express the gradients using matrix notation.

a) $f(x) = a^{\mathsf{T}}x + b$, where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$.

b) $f(x) = x^{\mathsf{T}}Ax$, for $A \in \mathbb{R}^{n \times n}$.

c) $f(x) = x^{\mathsf{T}}Ax$, where $A = A^{\mathsf{T}} \in \mathbb{R}^{n \times n}$. (Yes, this is a special case of the previous one.)

Solution.

a) Since $f(x) = a_1x_1 + \cdots + a_nx_n + b$, we have $\frac{\partial f_i}{\partial x_i} = a_i$, so $\nabla f(x) = a$. So the gradient of an affine function is constant, and equal to the vector associated with the linear part of the function.

b) We write out f explicitly as

$$f(x) = \sum_{i=1}^{n} x_i (Ax)_i = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j.$$

Therefore we have

$$\frac{\partial f}{\partial x_k} = \sum_{i=1}^{n} A_{ik} x_i + \sum_{j=1}^{n} A_{kj} x_j = (A^\mathsf{T} x)_k + (Ax)_k.$$

So $\nabla f(x) = (A + A^\mathsf{T})x$.

c) By the previous problem, we have $\nabla f(x) = 2Ax$.

2.210. Express the following statements in matrix language. You can assume that all matrices mentioned have appropriate dimensions. Here is an example: "Every column of C is a linear combination of the columns of B" can be expressed as "$C = BF$ for some matrix F".

There can be several answers; one is good enough for us.

a) Suppose Z has n columns. For each i, row i of Z is a linear combination of rows $i, \ldots, n$ of Y.

b) W is obtained from V by permuting adjacent odd and even columns (*i.e.*, 1 and 2, 3 and 4, ...).

c) Each column of P makes an acute angle with each column of Q.

d) Each column of P makes an acute angle with the corresponding column of Q.

e) The first k columns of A are orthogonal to the remaining columns of A.

Solution.

a) $Z = UY$, where U is upper triangular, *i.e.*, $U_{ij} = 0$ for $i > j$.

b) $W = VS$, where S is the odd-even switch matrix, defined as

$$S = \begin{bmatrix} e_2 & e_1 & e_4 & e_3 & \cdots & e_m & e_{m-1} \end{bmatrix}.$$

c) All entries of the matrix $P^\mathsf{T} Q$ are positive.

d) The diagonal entries of the matrix $P^\mathsf{T} Q$ are positive.

e) $A^\mathsf{T} A$ has the block diagonal form

$$A^\mathsf{T} A = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix},$$

where $B_{11} \in \mathbb{R}^{k \times k}$.