2.20. State equations for a linear mechanical system. The equations of motion of a lumped mechanical system undergoing small motions can be expressed as

\[ M \ddot{q} + D \dot{q} + Kq = f \]

where \( q(t) \in \mathbb{R}^k \) is the vector of deflections, \( M, D, \) and \( K \) are the mass, damping, and stiffness matrices, respectively, and \( f(t) \in \mathbb{R}^k \) is the vector of externally applied forces. Assuming \( M \) is invertible, write linear system equations for the mechanical system, with state

\[
x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix},
\]

input \( u = f \), and output \( y = q \).

2.100. A mass subject to applied forces. Consider a unit mass subject to a time-varying force \( f(t) \) for \( 0 \leq t \leq n \). Let the initial position and velocity of the mass both be zero. Suppose that the force has the form \( f(t) = x_j \) for \( j - 1 \leq t < j \) and \( j = 1, \ldots, n \). Let \( y_1 \) and \( y_2 \) denote, respectively, the position and velocity of the mass at time \( t = n \).

a) Find the matrix \( A \in \mathbb{R}^{2 \times n} \) such that \( y = Ax \).

b) For \( n = 4 \), find a sequence of input forces \( x_1, \ldots, x_n \) that moves the mass to position 1 with velocity 0 at time \( n \).

2.130. Most common symbol in a given position. Consider (again) the following Markov language. We have an alphabet of \( n \) symbols \( 1, 2, \ldots, n \). A sentence is a finite sequence of symbols, \( k_1, \ldots, k_m \) where \( k_i \in \{1, \ldots, n\} \). A language or code consists of a set of sequences, which we will call the allowable sequences. A language is called Markov if the allowed sequences can be described by giving the allowable transitions between consecutive symbols. For each symbol we give a set of symbols which are allowed to follow the symbol. As a simple example, consider a Markov language with three symbols \( 1, 2, 3 \). Symbol 1 can be followed by 1 or 3; symbol 2 must be followed by 3; and symbol 3 can be followed by 1 or 2. The sentence 1132313 is allowable (i.e., in the language); the sentence 1132312 is not allowable (i.e., not in the language). To describe the allowed symbol transitions we can define a matrix \( A \in \mathbb{R}^{n \times n} \) by

\[
A_{ij} = \begin{cases} 
1 & \text{if symbol } i \text{ is allowed to follow symbol } j \\
0 & \text{if symbol } i \text{ is not allowed to follow symbol } j
\end{cases}
\]

There are five symbols \( 1, 2, 3, 4, 5 \), and the following symbol transition rules:

- 1 must be followed by 2 or 3
- 2 must be followed by 2 or 5
- 3 must be followed by 1
• 4 must be followed by 4 or 2 or 5
• 5 must be followed by 1 or 3

Among all allowed sequences of length 10, find the most common value for the seventh symbol. In principle you could solve this problem by writing down all allowed sequences of length 10, and counting how many of these have symbol $i$ as the seventh symbol, for $i = 1, \ldots, 5$. (We’re interested in the symbol for which this count is largest.) But we’d like you to use a smarter approach. Explain clearly how you solve the problem, as well as giving the specific answer. Hint: you may find the interpretation of $A^k$ helpful.

2.160. Some matrices from signal processing. We consider $x \in \mathbb{R}^n$ as a signal, with $x_i$ the (scalar) value of the signal at (discrete) time period $i$, for $i = 1, \ldots, n$. Below we describe several transformations of the signal $x$, that produce a new signal $y$ (whose dimension varies). For each one, find a matrix $A$ for which $y = Ax$.

a) 2× up-conversion with linear interpolation. We take $y \in \mathbb{R}^{2n-1}$. For $i$ odd, $y_i = (x_{i+1})/2$. For $i$ even, $y_i = (x_i/2 + x_{i+1})/2$. Roughly speaking, this operation doubles the sample rate, inserting new samples in between the original ones using linear interpolation.

b) 2× down-sampling. We assume here that $n$ is even, and take $y \in \mathbb{R}^{n/2}$, with $y_i = x_{2i}$.

c) 2× down-sampling with averaging. We assume here that $n$ is even, and take $y \in \mathbb{R}^{n/2}$, with $y_i = (x_{2i-1} + x_{2i})/2$.

2.170. Affine functions. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called affine if for any $x, y \in \mathbb{R}^n$ and any $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1$, we have

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

(Without the restriction $\alpha + \beta = 1$, this would be the definition of linearity.)

a) Suppose that $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Show that the function $f(x) = Ax + b$ is affine.

b) Now the converse: Show that any affine function $f$ can be represented as $f(x) = Ax + b$, for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. (This representation is unique: for a given affine function $f$ there is only one $A$ and one $b$ for which $f(x) = Ax + b$ for all $x$.)

Hint. Show that the function $g(x) = f(x) - f(0)$ is linear.

You can think of an affine function as a linear function, plus an offset. In some contexts, affine functions are (mistakenly, or informally) called linear, even though in general they are not. (Example: $y = mx + b$ is described as ‘linear’ in US high schools.)