2.100. A mass subject to applied forces. Consider a unit mass subject to a time-varying force \( f(t) \) for \( 0 \leq t \leq n \). Let the initial position and velocity of the mass both be zero. Suppose that the force has the form \( f(t) = x_j \) for \( j - 1 \leq t < j \) and \( j = 1, \ldots, n \). Let \( y_1 \) and \( y_2 \) denote, respectively, the position and velocity of the mass at time \( t = n \).

a) Find the matrix \( A \in \mathbb{R}^{2 \times n} \) such that \( y = Ax \).

b) For \( n = 4 \), find a sequence of input forces \( x_1, \ldots, x_n \) that moves the mass to position 1 with velocity 0 at time \( n \).

2.120. Counting sequences in a language or code. We consider a language or code with an alphabet of \( n \) symbols \( 1, 2, \ldots, n \). A sentence is a finite sequence of symbols, \( k_1, \ldots, k_m \) where \( k_i \in \{1, \ldots, n\} \). A language or code consists of a set of sequences, which we will call the allowable sequences. A language is called Markov if the allowed sequences can be described by giving the allowable transitions between consecutive symbols. For each symbol we give a set of symbols which are allowed to follow the symbol. As a simple example, consider a Markov language with three symbols \( 1, 2, 3 \). Symbol 1 can be followed by 1 or 3; symbol 2 must be followed by 3; and symbol 3 can be followed by 1 or 2. The sentence \( 1132313 \) is allowable (i.e., in the language); the sentence \( 1132312 \) is not allowable (i.e., not in the language). To describe the allowed symbol transitions we can define a matrix \( A \in \mathbb{R}^{n \times n} \) by

\[
A_{ij} = \begin{cases} 1 & \text{if symbol } i \text{ is allowed to follow symbol } j \\ 0 & \text{if symbol } i \text{ is not allowed to follow symbol } j \end{cases}
\]

a) Let \( B = A^r \). Give an interpretation of \( B_{ij} \) in terms of the language.

b) Consider the Markov language with five symbols \( 1, 2, 3, 4, 5 \), and the following transition rules:

- 1 must be followed by 2 or 3
- 2 must be followed by 2 or 5
- 3 must be followed by 1
- 4 must be followed by 4 or 2 or 5
- 5 must be followed by 1 or 3

Find the total number of allowed sentences of length 10. Compare this number to the simple code that consists of all sequences from the alphabet (i.e., all symbol transitions are allowed). In addition to giving the answer, you must explain how you solve the problem. Do not hesitate to use Julia.
2.170. **Affine functions.** A function \( f : \mathbb{R}^n \to \mathbb{R}^m \) is called **affine** if for any \( x, y \in \mathbb{R}^n \) and any \( \alpha, \beta \in \mathbb{R} \) with \( \alpha + \beta = 1 \), we have

\[
f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).
\]

(Without the restriction \( \alpha + \beta = 1 \), this would be the definition of linearity.)

a) Suppose that \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). Show that the function \( f(x) = Ax + b \) is affine.

b) Now the converse: Show that any affine function \( f \) can be represented as \( f(x) = Ax + b \), for some \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). (This representation is unique: for a given affine function \( f \) there is only one \( A \) and one \( b \) for which \( f(x) = Ax + b \) for all \( x \).)

*Hint.* Show that the function \( g(x) = f(x) - f(0) \) is linear.

You can think of an affine function as a linear function, plus an offset. In some contexts, affine functions are (mistakenly, or informally) called linear, even though in general they are not. (Example: \( y = mx + b \) is described as ‘linear’ in US high schools.)

2.210. **Express the following statements in matrix language.** You can assume that all matrices mentioned have appropriate dimensions. Here is an example: “Every column of \( C \) is a linear combination of the columns of \( B \)” can be expressed as “\( C = BF \) for some matrix \( F \).”

There can be several answers; one is good enough for us.

a) Suppose \( Z \) has \( n \) columns. For each \( i \), row \( i \) of \( Z \) is a linear combination of rows \( i, \ldots, n \) of \( Y \).

b) \( W \) is obtained from \( V \) by permuting adjacent odd and even columns (i.e., 1 and 2, 3 and 4, \ldots).

c) Each column of \( P \) makes an acute angle with each column of \( Q \).

d) Each column of \( P \) makes an acute angle with the corresponding column of \( Q \).

e) The first \( k \) columns of \( A \) are orthogonal to the remaining columns of \( A \).

2.230. **Population dynamics.** An ecosystem consists of \( n \) species that interact (say, by eating other species, eating each other’s food sources, eating each other’s predators, and so on). We let \( x(t) \in \mathbb{R}^n \) be the vector of deviations of the species populations (say, in thousands) from some equilibrium values (which don’t matter here), in time period (say, month) \( t \). In this model, time will take on the discrete values \( t = 0, 1, 2, \ldots \). Thus \( x_3(4) < 0 \) means that the population of species 3 in time period 4 is below its equilibrium level. (It does not mean the population of species 3 is negative in time period 4.) The population (deviations) follows a discrete-time linear dynamical system:

\[
x(t + 1) = Ax(t).
\]

We refer to \( x(0) \) as the **initial population perturbation**.
The questions below pertain to the specific case with $n = 10$ species, with matrix $A$ given in `pop_dyn_data.json`.

a) Suppose the initial perturbation is $x(0) = e_4$ (meaning, we inject one thousand new creatures of species 4 into the ecosystem at $t = 0$). How long will it take to affect the other species populations? In other words, report a vector $s$, where $s_i$ is the smallest $t$ for which $x_i(t) \neq 0$. (We have $s_4 = 0$).

b) Population control. We can choose any initial perturbation that satisfies $|x_i(0)| \leq 1$ for each $i = 1, \ldots, 10$. (We achieve this by introducing additional creatures and/or hunting and fishing.) What initial perturbation $x(0)$ would you choose in order to maximize the population of species 1 at time $t = 10$? Explain your reasoning. Give the initial perturbation, and using your selected initial perturbation, give $x_1(10)$ and plot $x_1(t)$ versus $t$ for $t = 0, \ldots, 40$. 

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