EE263 Midterm Exam — Solutions

Summer 2026

This key gives the correct choice(s) for each of the 20 multiple-choice questions, with a brief justification. Questions marked *Select all that apply.* may have any number of correct choices.

1. Linearization of a nonlinear function. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x_1, x_2) = \begin{bmatrix} x_1^2 + x_2 \\ x_1 x_2 \end{bmatrix}$. Let L be the linearization of f around the point $(2, 1)$. What is $L(3, 0)$?

(a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 8 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 8 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Answer: (b). $f(2, 1) = (5, 2)$, and the Jacobian at $(2, 1)$ is $J = \begin{bmatrix} 2x_1 & 1 \\ x_2 & x_1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$. Hence $L(3, 0) = f(2, 1) + J\big((3, 0) - (2, 1)\big) = (5, 2) + \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (5, 2) + (3, -1) = (8, 1)$.

2. A random matrix. We use a random number generator to populate the entries of a matrix $A \in \mathbb{R}^{1000 \times 10{,}000}$. With very high probability, A is ________________ and the angle between any two columns of A is ________________.

(a) not full-rank – uniformly distributed between 0° and 180°

(b) full-rank – uniformly distributed between 0° and 180°

(c) not full-rank – very close to 90°

(d) full-rank – very close to 90°

Answer: (d). A random $1000 \times 10{,}000$ matrix has full rank ($= 1000$) with probability essentially 1. In high dimensions two independent random vectors are nearly orthogonal, so the angle between any two columns concentrates near 90°.

3. Changing coordinates. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ have matrix A in the standard basis. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be another basis, and let $P = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$, where each v_i is written in standard coordinates. Let B be the matrix of T in the basis $\mathcal{B}$. Which of the following statements are correct? *Select all that apply.*

(a) $B = P^{-1}AP$.

(b) A and B have the same rank.

(c) A is invertible if and only if B is invertible.

(d) By choosing $\mathcal{B}$ appropriately, any invertible matrix A can be transformed into I.

Answer: (a), (b), (c). $B = P^{-1}AP$ is the change-of-basis (similarity) formula, so (a) holds; similar matrices have equal rank and are invertible together, so (b) and (c) hold. (d) is false: A is similar to I only if $A = PIP^{-1} = I$.

4. **Right inverses of a fat matrix.** Let $A \in \mathbb{R}^{m\times n}$ with $m < n$. Which statement about the number of right inverses of A is correct?

(a) A either has no right inverse or has infinitely many right inverses.

(b) A may have no right inverse, exactly one right inverse, or infinitely many right inverses.

(c) If A has a right inverse, then that right inverse is unique.

(d) A may have a finite number greater than one of right inverses.

Answer: (a). With $m < n$: if A has full row rank it has infinitely many right inverses (a particular one plus anything mapping into $\text{null}(A) \neq \{0\}$); otherwise it has none. A unique right inverse is impossible, so (a).

5. **Counting walks in a graph.** An undirected graph with nodes $1, \ldots, 6$ has adjacency matrix A, where $A_{ij} = \begin{cases} 1, & \text{if nodes } i \text{ and } j \text{ are connected by an edge,} \\ 0, & \text{otherwise.} \end{cases}$ The graph has no self-loops, so $A_{ii} = 0$ for all i. Its adjacency matrix is

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

How many walks of length 20 are there from node 1 to node 2?

Hint: Recall the interpretation of the entries of A^k from homework 1.

(a) 0

(b) 3^{10}

(c) 3^{19}

(d) 3^{20}

Answer: (c). $(A^k)_{ij}$ counts walks of length k from i to j. Here $A = \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix}$ with J the 3×3 all-ones matrix and $J^2 = 3J$, so $A^2 = \begin{bmatrix} 3J & 0 \\ 0 & 3J \end{bmatrix}$ and $A^{20} = 3^{19} \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$. Nodes 1 and 2 lie in the same block, so $(A^{20})_{12} = 3^{19}$.

6. **Advertising and product sales.** A company sells cola, lemon soda, and bottled water, indexed by 1, 2, and 3, respectively. Let x_i be the change in advertising spending for product i, and let y_i be the resulting change in sales. For small changes, $y = Ax$. Advertising is generally intended to increase sales of the advertised product, but it may also draw customers away from competing products. Cola and lemon soda are close competitors, whereas bottled water competes less directly with cola. *Select all that apply.*

   (a) $a_{ii} > 0$ for all i

   (b) $a_{21} < a_{31} < 0$

   (c) $a_{11} + a_{21} < 0$

   (d) $a_{31} > 0$

   Answer: (a), (b). Own advertising raises own sales, so $a_{ii} > 0$ (a). Cola ads draw customers from the close competitor (lemon soda) more strongly than from water, and both effects are negative, so $a_{21} < a_{31} < 0$ (b). (c) need not hold (the own-effect may dominate the cannibalization), and (d) has the wrong sign since $a_{31} \leq 0$.

7. **Linear functions.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function, and $U = \{u_1, \cdots, u_k\} \subset \mathbb{R}^n$. We construct $V = \{v_1, \cdots, v_k\}$, where $v_i = f(u_i)$. Which of the following statements is correct?

   (a) If U is an independent set of vectors, then so is V.

   (b) If V is an independent set of vectors, then so is U.

   (c) If U is an independent set and $k \geq m$, then any $y = f(x)$ can be represented as a linear combination of v_i's.

   (d) If any $y = f(x)$ can be represented as a linear combination of v_i's and $k < m$, then V has to be an independent set.

   Answer: (b). If $V = \{f(u_i)\}$ is independent, then U must be independent: a dependence $\sum c_i u_i = 0$ would give $\sum c_i v_i = f(0) = 0$. So (b). (a) is false (a linear map can collapse independent vectors onto dependent ones); (c) and (d) fail in general.

8. **Union of subspaces.** Let S and T be subspaces of $\mathbb{R}^n$, such that $S \cup T = \mathbb{R}^n$. Which of the following conclusions is correct?

   (a) At least one of the two subspaces must be $\mathbb{R}^n$.

   (b) $\dim S + \dim T > n$.

   (c) Every $x \in \mathbb{R}^n$ has a unique representation $x = s + t$, where $s \in S$ and $t \in T$.

   (d) It is possible that none of the two subspaces is a subset of the other one.

   Answer: (a). If $S \cup T = \mathbb{R}^n$, one of them must be all of $\mathbb{R}^n$. Otherwise pick $s \in S \setminus T$ and $t \in T \setminus S$; then $s + t$ lies in neither S nor T, contradicting $S \cup T = \mathbb{R}^n$. Hence (a); (b), (c), (d) are false.

9. Left inverse. If A is defined as below, which of the following statements is true? *Select all that apply.*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 1 \\ 1 & 4 & 5 \end{bmatrix}$$

(a) There exists a matrix B such that $BA = I$.

(b) There exists a matrix B such that $BA = I$ and B is lower-triangular.

(c) There exists a matrix B such that $BA = I$ and B has rank 2.

(d) There exists a matrix B such that $AB = I$.

Answer: (a), (b). A (5×3) has full column rank, so a left inverse exists (a). One explicit choice is $B = \begin{bmatrix} I_3 & 0 \end{bmatrix}$, which is lower-triangular (b). (c) is impossible: $BA = I_3$ forces $\operatorname{rank} B \geq 3$. (d) is impossible: A is tall with $\operatorname{rank} A = 3 < 5$, so it has no right inverse.

10. Orthonormal columns. Let $U \in \mathbb{R}^{m \times n}$ be a skinny matrix, with $m > n$, whose columns are orthonormal. Which of the following statements about UU^T are correct? *Select all that apply.*

(a) $\operatorname{rank}(UU^\mathsf{T}) = \operatorname{rank}(U) = n$

(b) $UU^\mathsf{T} \neq I_m$

(c) There exists a subspace $S \subseteq \mathbb{R}^m$ with $\dim(S) > 0$ such that $UU^\mathsf{T}x = x$ for every $x \in S$

(d) $UU^\mathsf{T}x = 0$ only when $x = 0$

Answer: (a), (b), (c). UU^T is the orthogonal projector onto $\operatorname{range}(U)$. Its rank is $\operatorname{rank}(U) = n$ (a); it is not I_m since $n < m$ (b); and it fixes every vector of $\operatorname{range}(U)$, a subspace of dimension $n > 0$ (c). (d) is false: its nullspace $\operatorname{range}(U)^\perp$ has dimension $m - n > 0$.

11. Projection matrix (I). A square matrix P is called a *projection matrix* if $P = P^\mathsf{T}$ and $P^2 = P$. Let P and Q be projection matrices. Which of the following statements are correct? *Select all that apply.*

(a) $I - P$ and $I - Q$ are projection matrices.

(b) PQ is a projection matrix.

(c) If P and Q are full-rank, then $P = Q$.

(d) If $u \in \mathbb{R}^n$ and $\|u\| = 1$, then $I - uu^\mathsf{T}$ is a projection with rank $n - 1$.

Answer: (a), (c), (d). If P is a projection, so is $I - P$ (a). A full-rank projection equals I, so two full-rank projections are both I, hence equal (c). For $\|u\| = 1$, $I - uu^\mathsf{T}$ is a projection of rank $n - 1$ (d). (b) is false: PQ is a projection only when P and Q commute.

12. Projection matrix (II). A square matrix P is called a *projection matrix* if $P = P^\mathsf{T}$ and $P^2 = P$. For a given projection matrix P and a given vector $x \in \mathbb{R}^n$, which of the following statements is correct?

(a) $\|Px\|$ can be larger than $\|x\|$.

(b) If $\|Px\| = \|x\|$, then $P = I$.

(c) If $\|Px\| = 0$, then $x \in \mathrm{range}(P)^\perp$.

(d) Some P and x exist such that $\|Px\| < \|x\|$ and $x \in \mathrm{range}(P)$.

Answer: (c). $\|Px\| = 0 \iff Px = 0 \iff x \in \mathrm{null}(P) = \mathrm{range}(P)^\perp$, giving (c). (a) is false since $\|Px\| \leq \|x\|$; (b) is false ($\|Px\| = \|x\|$ iff $x \in \mathrm{range}(P)$, not $P = I$); (d) is false since $x \in \mathrm{range}(P) \Rightarrow Px = x \Rightarrow \|Px\| = \|x\|$.

13. Rank of a product. Let A, B be 10×10 matrices such that $\mathrm{rank}(A) = 9$ and $\mathrm{rank}(B) = 7$. How many possible values exist for $\mathrm{rank}(AB)$?

(a) 2

(b) 3

(c) 5

(d) 7

Answer: (a). By Sylvester's inequality, $\mathrm{rank}\,A + \mathrm{rank}\,B - n \leq \mathrm{rank}(AB) \leq \min(\mathrm{rank}\,A, \mathrm{rank}\,B)$, i.e. $6 \leq \mathrm{rank}(AB) \leq 7$. Both values are attainable, so there are 2 possible values.

14. Orthogonal transformations in $\mathbb{R}^2$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with matrix A, and suppose that $A^\mathsf{T} A = I$. Which of the following statements are correct? *Select all that apply.*

(a) T preserves the lengths of vectors, and the area of every region.

(b) T may be neither a rotation nor a reflection.

(c) Some $k > 1$ exists such that $A^k = A$.

(d) If T is obtained by first applying a rotation and then a reflection, then T is a reflection across some line through the origin.

Answer: (a), (d). An orthogonal T preserves lengths and areas (a). A rotation followed by a reflection has $\det = -1$, hence is a reflection across a line (d). (b) is false: every 2×2 orthogonal map is a rotation or a reflection. (c) is false: $A^k = A$ needs $A^{k-1} = I$, which fails for a rotation by an irrational multiple of π (infinite order).

15. Sum of subspaces. Let S and T be subspaces of $\mathbb{R}^n$. Define $S+T = \{s+t : s \in S,\ t \in T\}$. Which of the following statements are correct? *Select all that apply.*

Hint: It might be helpful to think of a basis for S, extend it to a basis for $S+T$, and then extend it to a basis for $\mathbb{R}^n$.

(a) If $\dim(S) + \dim(T) = n$, then $S + T = \mathbb{R}^n$.

(b) $\dim(S+T) = \dim(S) + \dim(T) - \dim(S \cap T)$

(c) $(S \cap T)^\perp = S^\perp + T^\perp$

(d) If $S + T = \mathbb{R}^n$, then every $x \in \mathbb{R}^n$ can be written uniquely as $x = s + t$, where $s \in S$ and $t \in T$.

Answer: (b), (c). The dimension formula $\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$ always holds (b), and $(S \cap T)^\perp = S^\perp + T^\perp$ always holds (c). (a) is false unless $S \cap T = \{0\}$; (d) is false, since a unique decomposition $x = s + t$ requires $S \cap T = \{0\}$ (a direct sum).

16. Range and null space. Let $A \in \mathbb{R}^{n\times p}$ and $B \in \mathbb{R}^{n\times q}$. Which of the following statements are correct? *Select all that apply.*

(a) If $\operatorname{rank}\big(\begin{bmatrix} A & B \end{bmatrix}\big) = \operatorname{rank}(A) = \operatorname{rank}(B)$, then $\operatorname{range}(A) = \operatorname{range}(B)$.

(b) If $\operatorname{rank}\big(\begin{bmatrix} A & B \end{bmatrix}\big) = \operatorname{rank}(A) = \operatorname{rank}(B)$ and $p = q$, then $\operatorname{null}(A) = \operatorname{null}(B)$.

(c) $x \in \operatorname{null}(A^\mathsf{T})$ if and only if $x \notin \operatorname{range}(A)$.

(d) If $\operatorname{null}(A) = \{0\}$ and $\operatorname{range}(A) \subseteq \operatorname{range}(B)$, then $p \le q$.

Answer: (a), (d). $\operatorname{rank}\begin{bmatrix} A & B \end{bmatrix} = \dim(\operatorname{range} A + \operatorname{range} B)$; if it equals both $\operatorname{rank} A$ and $\operatorname{rank} B$, then $\operatorname{range} A = \operatorname{range} B$ (a). If $\operatorname{null}(A) = \{0\}$ then $p = \operatorname{rank} A \le \operatorname{rank} B \le q$ (d). (b) is false (equal column spaces need not give equal null spaces); (c) is false, since $\operatorname{null}(A^\mathsf{T}) = \operatorname{range}(A)^\perp$, not the set-complement of $\operatorname{range}(A)$.

17. Hadamard matrices. Consider the set of matrices constructed as follows:

$$H_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad H_n = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix} \quad \text{for } n > 2.$$

Thus, H_n is a $2^{n-1} \times 2^{n-1}$ matrix. Which of the following statements are true about the matrices H_n, $M = \begin{bmatrix} H_n \\ I \end{bmatrix}$, $N = \begin{bmatrix} H_n & I \end{bmatrix}$? *Select all that apply.*

(a) H_n is an orthogonal matrix.

(b) $\dim\big(\operatorname{null}(N)\big) = n$.

(c) M has a right inverse.

(d) $NN^\mathsf{T} = 2I$.

Answer: (a), (d). H_n is orthogonal, by induction using the $1/\sqrt{2}$ scaling (a). For $N = \begin{bmatrix} H_n & I \end{bmatrix}$, $NN^\mathsf{T} = H_n H_n^\mathsf{T} + I = 2I$ (d). (b) is false: $\dim \operatorname{null}(N) = 2^{n-1}$, not n. (c) is false: $M = \begin{bmatrix} H_n \\ I \end{bmatrix}$ is tall, so it has a left inverse, not a right inverse.

18. QR decomposition (I). We apply the Gram-Schmidt procedure to the vectors $a_1, \dots, a_m \in \mathbb{R}^n$. The algorithm outputs matrices $Q \in \mathbb{R}^{n\times r}$ and $R \in \mathbb{R}^{r\times m}$. Which of the following statements are correct? *Select all that apply.*

(a) If we change the order of a_i's, the resulting Q will have the same columns, but in a different order.

(b) If we change the order of a_i's, the resulting R will have the same columns, but in a different order.

(c) If we apply the procedure to $\{a_1, a_1+a_2, \cdots, a_1+\cdots+a_m\}$, the resulting Q will change.

(d) If we apply the procedure to $\{a_1, a_1+a_2, \cdots, a_1+\cdots+a_m\}$, the resulting R will not change.

Answer: none of the four (leave all unmarked). All four statements are false, so the correct response is to mark none of them. Gram–Schmidt is order-dependent: reordering the a_i changes the *columns* of Q (not merely their order) and changes R, so (a) and (b) are false. Applying the procedure to the partial sums $\{a_1,\ a_1+a_2,\ \dots\}$ leaves every nested span $\operatorname{span}\{a_1,\dots,a_j\}$ unchanged, so Q is unchanged, making (c) false; while R does change, making (d) false.

19. QR decomposition (II). Let $A \in \mathbb{R}^{m\times n}$ be a matrix of rank r. We construct $C = \begin{bmatrix} A & B \end{bmatrix}$, where B is an $m \times k$ matrix. We then apply the Gram-Schmidt procedure to C and get the matrices Q and R such that $C = QR$. Which of the following statements are correct? *Select all that apply.*

(a) If Q has $n+k$ columns, then R is a square, invertible, and upper-triangular matrix.

(b) If Q has r columns, then R cannot be a square matrix.

(c) If $\operatorname{range}(Q) = \mathbb{R}^m$, then $\operatorname{range}(A) \cup \operatorname{range}(B) = \mathbb{R}^n$.

(d) If R is square and invertible, then $\operatorname{range}(A) \cap \operatorname{range}(B) = \{0\}$.

Answer: (a), (b), (d). (a): if Q has $n+k$ columns, all columns of $\begin{bmatrix} A & B \end{bmatrix}$ are independent, so R is $(n+k)\times(n+k)$, upper-triangular, and invertible. (b): if Q has $r = \operatorname{rank} A$ columns, R is $r \times (n+k)$ with $r \le n < n+k$, so R is never square. (d): R square and invertible means $\begin{bmatrix} A & B \end{bmatrix}$ has independent columns, so $\operatorname{range} A \cap \operatorname{range} B = \{0\}$. (c) is false: $\operatorname{range}(Q) = \mathbb{R}^m$ gives $\operatorname{range} A + \operatorname{range} B = \mathbb{R}^m$ (a sum, equal to $\mathbb{R}^m$, not $\mathbb{R}^n$).

20. A failed sensor. A system has 8 unknown inputs and 32 sensors. Under normal operation, the measurements satisfy $y = Ax$, where $A \in \mathbb{R}^{32\times 8}$. The sensor settings are such that any 8 rows of A are linearly independent. We know that exactly one entry of y is corrupted and differs from its correct value, but we don't know which sensor that is. A Julia program may split or append matrices and vectors and use `rank` to test consistency. Once enough accurate measurements have been identified, solving for x does not require an additional call to `rank`.

What are, respectively, (1) the minimum number of calls to `rank` needed to guarantee that the correct value of x can be recovered, and (2) the minimum number of calls to `rank` needed to guarantee that the corrupted measurement can be identified?

(a) 1 and 5

(b) 2 and 5

(c) 1 and 6

(d) 2 and 6

Answer: (a) — 1 and 5. Recovering x takes 1 call: test any 9 sensors for consistency; if consistent, all 9 are correct (solve x from any 8); if not, the faulty sensor is among those 9, so the other 23 are correct. Identifying the faulty sensor takes 5 calls: whether the fault lies in a given set is testable with one `rank` call (test the set if it has ≥ 9 sensors, else its complement), so binary search over 32 sensors needs $\lceil \log_2 32 \rceil = 5$ calls.