EE263 Midterm Exam
Summer 2026

Name: ____________________________ **SUNet ID:** ____________________

The Stanford Honor Code

The Honor Code is an undertaking of the Stanford academic community, individually and collectively. Its purpose is to uphold a culture of academic honesty.

Students will support this culture of academic honesty by neither giving nor accepting unpermitted academic aid in any work that serves as a component of grading or evaluation, including assignments, examinations, and research.

Instructors will support this culture of academic honesty by providing clear guidance, both in their course syllabi and in response to student questions, on what constitutes permitted and unpermitted aid. Instructors will also not take unusual or unreasonable precautions to prevent academic dishonesty.

Students and instructors will also cultivate an environment conducive to academic integrity. While instructors alone set academic requirements, the Honor Code is a community undertaking that requires students and instructors to work together to ensure conditions that support academic integrity.

I acknowledge and accept the Stanford Honor Code. I affirm that I have neither given nor received unpermitted aid on this examination.

Signature: _________________________________ **Date:** __________________

Instructions

- This is a closed-book, closed-notes exam. You may use one double-sided sheet of notes. No calculators, phones, or other electronic devices are allowed.
- The exam consists of 20 multiple-choice questions. Most questions have exactly one correct answer. Questions that may have more than one correct answer are marked with the instruction *Select all that apply.*; for such a question, any number of the four choices may be correct, including none of them and all of them.
- Each question is worth 4 points. For a question with exactly one correct answer, all 4 points are awarded for selecting the correct answer and no other answer. For a question marked *Select all that apply.*, each of the four choices is worth 1 point: you receive the point for a choice if you correctly mark it when it is true or correctly leave it unmarked when it is false.
- We have made every effort to state the questions clearly. If you believe that a question is unclear or ambiguous, you may make a reasonable assumption that resolves the ambiguity and solve the problem using that assumption. Briefly state your assumption and, if helpful, give a short explanation of your reasoning. If your interpretation is reasonable, it may receive credit even if it differs from the interpretation we intended.
- Explanations are not required in general. A brief explanation is useful only when you are making an assumption, identifying an ambiguity, or using an interpretation that may not be obvious from your selected answer.
- If you believe that a question contains an error, you are welcome to ask for clarification. If a question is found to be incorrect because of an error in the exam, full credit for that question will be awarded to everyone. You may therefore skip such a question, or state a reasonable correction or assumption and solve the resulting problem.
- Clearly mark your final answer or answers for each question. If you change an answer, make your final selection unambiguous.
- You have 90 minutes.

1. **Linearization of a nonlinear function.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x_1, x_2) = \begin{bmatrix} x_1^2 + x_2 \\ x_1 x_2 \end{bmatrix}$. Let L be the linearization of f around the point $(2, 1)$. What is $L(3, 0)$?

   - ☐ $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
   - ☐ $\begin{bmatrix} 8 \\ 1 \end{bmatrix}$
   - ☐ $\begin{bmatrix} 8 \\ 3 \end{bmatrix}$
   - ☐ $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

2. **A random matrix.** We use a random number generator to populate the entries of a matrix $A \in \mathbb{R}^{1000 \times 10{,}000}$. With very high probability, A is ________________ and the angle between any two columns of A is ________________.

   - ☐ not full-rank – uniformly distributed between 0° and 180°
   - ☐ full-rank – uniformly distributed between 0° and 180°
   - ☐ not full-rank – very close to 90°
   - ☐ full-rank – very close to 90°

3. Changing coordinates. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ have matrix A in the standard basis. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be another basis, and let $P = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$, where each v_i is written in standard coordinates. Let B be the matrix of T in the basis $\mathcal{B}$. Which of the following statements are correct? *Select all that apply.*

- ☐ $B = P^{-1}AP$.
- ☐ A and B have the same rank.
- ☐ A is invertible if and only if B is invertible.
- ☐ By choosing $\mathcal{B}$ appropriately, any invertible matrix A can be transformed into I.

4. Right inverses of a fat matrix. Let $A \in \mathbb{R}^{m \times n}$ with $m < n$. Which statement about the number of right inverses of A is correct?

- ☐ A either has no right inverse or has infinitely many right inverses.
- ☐ A may have no right inverse, exactly one right inverse, or infinitely many right inverses.
- ☐ If A has a right inverse, then that right inverse is unique.
- ☐ A may have a finite number greater than one of right inverses.

5. Counting walks in a graph. An undirected graph with nodes $1, \ldots, 6$ has adjacency matrix A, where $A_{ij} = \begin{cases} 1, & \text{if nodes } i \text{ and } j \text{ are connected by an edge,} \\ 0, & \text{otherwise.} \end{cases}$ The graph has no self-loops, so $A_{ii} = 0$ for all i. Its adjacency matrix is

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

How many walks of length 20 are there from node 1 to node 2?
Hint: Recall the interpretation of the entries of A^k from homework 1.

- ☐ 0
- ☐ 3^{10}
- ☐ 3^{19}
- ☐ 3^{20}

6. Advertising and product sales. A company sells cola, lemon soda, and bottled water, indexed by 1, 2, and 3, respectively. Let x_i be the change in advertising spending for product i, and let y_i be the resulting change in sales. For small changes, $y = Ax$. Advertising is generally intended to increase sales of the advertised product, but it may also draw customers away from competing products. Cola and lemon soda are close competitors, whereas bottled water competes less directly with cola. *Select all that apply.*

- ☐ $a_{ii} > 0$ for all i
- ☐ $a_{21} < a_{31} < 0$
- ☐ $a_{11} + a_{21} < 0$
- ☐ $a_{31} > 0$

7. Linear functions. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function, and $U = \{u_1, \cdots, u_k\} \subset \mathbb{R}^n$. We construct $V = \{v_1, \cdots, v_k\}$, where $v_i = f(u_i)$. Which of the following statements is correct?

- ☐ If U is an independent set of vectors, then so is V.
- ☐ If V is an independent set of vectors, then so is U.
- ☐ If U is an independent set and $k \geq m$, then any $y = f(x)$ can be represented as a linear combination of v_i's.
- ☐ If any $y = f(x)$ can be represented as a linear combination of v_i's and $k < m$, then V has to be an independent set.

8. Union of subspaces. Let S and T be subspaces of $\mathbb{R}^n$, such that $S \cup T = \mathbb{R}^n$. Which of the following conclusions is correct?

- ☐ At least one of the two subspaces must be $\mathbb{R}^n$.
- ☐ $\dim S + \dim T > n$.
- ☐ Every $x \in \mathbb{R}^n$ has a unique representation $x = s + t$, where $s \in S$ and $t \in T$.
- ☐ It is possible that none of the two subspaces is a subset of the other one.

9. Left inverse. If A is defined as below, which of the following statements is true? *Select all that apply.*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 1 \\ 1 & 4 & 5 \end{bmatrix}$$

- ☐ There exists a matrix B such that $BA = I$.
- ☐ There exists a matrix B such that $BA = I$ and B is lower-triangular.
- ☐ There exists a matrix B such that $BA = I$ and B has rank 2.
- ☐ There exists a matrix B such that $AB = I$.

10. Orthonormal columns. Let $U \in \mathbb{R}^{m \times n}$ be a skinny matrix, with $m > n$, whose columns are orthonormal. Which of the following statements about UU^T are correct? *Select all that apply.*

- ☐ $\operatorname{rank}(UU^\mathsf{T}) = \operatorname{rank}(U) = n$
- ☐ $UU^\mathsf{T} \neq I_m$
- ☐ There exists a subspace $S \subseteq \mathbb{R}^m$ with $\dim(S) > 0$ such that $UU^\mathsf{T}x = x$ for every $x \in S$
- ☐ $UU^\mathsf{T}x = 0$ only when $x = 0$

11. Projection matrix (I). A square matrix P is called a *projection matrix* if $P = P^\mathsf{T}$ and $P^2 = P$. Let P and Q be projection matrices. Which of the following statements are correct? <u>*Select all that apply.*</u>

- ☐ $I - P$ and $I - Q$ are projection matrices.
- ☐ PQ is a projection matrix.
- ☐ If P and Q are full-rank, then $P = Q$.
- ☐ If $u \in \mathbb{R}^n$ and $\|u\| = 1$, then $I - uu^\mathsf{T}$ is a projection with rank $n - 1$.

12. Projection matrix (II). A square matrix P is called a *projection matrix* if $P = P^\mathsf{T}$ and $P^2 = P$. For a given projection matrix P and a given vector $x \in \mathbb{R}^n$, which of the following statements is correct?

- ☐ $\|Px\|$ can be larger than $\|x\|$.
- ☐ If $\|Px\| = \|x\|$, then $P = I$.
- ☐ If $\|Px\| = 0$, then $x \in \text{range}(P)^\perp$.
- ☐ Some P and x exist such that $\|Px\| < \|x\|$ and $x \in \text{range}(P)$.

13. Rank of a product. Let A, B be 10×10 matrices such that $\text{rank}(A) = 9$ and $\text{rank}(B) = 7$. How many possible values exist for $\text{rank}(AB)$?

- ☐ 2
- ☐ 3
- ☐ 5
- ☐ 7

14. Orthogonal transformations in $\mathbb{R}^2$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with matrix A, and suppose that $A^\mathsf{T} A = I$. Which of the following statements are correct? *Select all that apply.*

- ☐ T preserves the lengths of vectors, and the area of every region.
- ☐ T may be neither a rotation nor a reflection.
- ☐ Some $k > 1$ exists such that $A^k = A$.
- ☐ If T is obtained by first applying a rotation and then a reflection, then T is a reflection across some line through the origin.

15. Sum of subspaces. Let S and T be subspaces of $\mathbb{R}^n$. Define $S+T = \{s+t : s \in S,\ t \in T\}$. Which of the following statements are correct? *Select all that apply.*
Hint: It might be helpful to think of a basis for S, extend it to a basis for $S+T$, and then extend it to a basis for $\mathbb{R}^n$.

- ☐ If $\dim(S) + \dim(T) = n$, then $S + T = \mathbb{R}^n$.
- ☐ $\dim(S + T) = \dim(S) + \dim(T) - \dim(S \cap T)$
- ☐ $(S \cap T)^\perp = S^\perp + T^\perp$
- ☐ If $S + T = \mathbb{R}^n$, then every $x \in \mathbb{R}^n$ can be written uniquely as $x = s + t$, where $s \in S$ and $t \in T$.

16. Range and null space. Let $A \in \mathbb{R}^{n\times p}$ and $B \in \mathbb{R}^{n\times q}$. Which of the following statements are correct? *Select all that apply.*

- ☐ If $\mathrm{rank}\big(\begin{bmatrix} A & B \end{bmatrix}\big) = \mathrm{rank}(A) = \mathrm{rank}(B)$, then $\mathrm{range}(A) = \mathrm{range}(B)$.
- ☐ If $\mathrm{rank}\big(\begin{bmatrix} A & B \end{bmatrix}\big) = \mathrm{rank}(A) = \mathrm{rank}(B)$ and $p = q$, then $\mathrm{null}(A) = \mathrm{null}(B)$.
- ☐ $x \in \mathrm{null}(A^\mathsf{T})$ if and only if $x \notin \mathrm{range}(A)$.
- ☐ If $\mathrm{null}(A) = \{0\}$ and $\mathrm{range}(A) \subseteq \mathrm{range}(B)$, then $p \le q$.

17. Hadamard matrices. Consider the set of matrices constructed as follows:

$$H_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad H_n = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix} \quad \text{for } n > 2.$$

Thus, H_n is a $2^{n-1} \times 2^{n-1}$ matrix. Which of the following statements are true about the matrices H_n, $M = \begin{bmatrix} H_n \\ I \end{bmatrix}$, $N = \begin{bmatrix} H_n & I \end{bmatrix}$? <u>*Select all that apply.*</u>

- ☐ H_n is an orthogonal matrix.
- ☐ $\dim\big(\text{null}(N)\big) = n$.
- ☐ M has a right inverse.
- ☐ $NN^{\mathsf{T}} = 2I$.

18. QR decomposition (I). We apply the Gram-Schmidt procedure to the vectors $a_1, \ldots, a_m \in \mathbb{R}^n$. The algorithm outputs matrices $Q \in \mathbb{R}^{n\times r}$ and $R \in \mathbb{R}^{r\times m}$. Which of the following statements are correct? <u>*Select all that apply.*</u>

- ☐ If we change the order of a_i's, the resulting Q will have the same columns, but in a different order.
- ☐ If we change the order of a_i's, the resulting R will have the same columns, but in a different order.
- ☐ If we apply the procedure to $\{a_1, a_1+a_2, \cdots, a_1+\cdots+a_m\}$, the resulting Q will change.
- ☐ If we apply the procedure to $\{a_1, a_1+a_2, \cdots, a_1+\cdots+a_m\}$, the resulting R will not change.

19. QR decomposition (II). Let $A \in \mathbb{R}^{m \times n}$ be a matrix of rank r. We construct $C = \begin{bmatrix} A & B \end{bmatrix}$, where B is an $m \times k$ matrix. We then apply the Gram-Schmidt procedure to C and get the matrices Q and R such that $C = QR$. Which of the following statements are correct? *Select all that apply.*

- ☐ If Q has $n + k$ columns, then R is a square, invertible, and upper-triangular matrix.
- ☐ If Q has r columns, then R cannot be a square matrix.
- ☐ If $\text{range}(Q) = \mathbb{R}^m$, then $\text{range}(A) \cup \text{range}(B) = \mathbb{R}^n$.
- ☐ If R is square and invertible, then $\text{range}(A) \cap \text{range}(B) = \{0\}$.

20. A failed sensor. A system has 8 unknown inputs and 32 sensors. Under normal operation, the measurements satisfy $y = Ax$, where $A \in \mathbb{R}^{32 \times 8}$. The sensor settings are such that any 8 rows of A are linearly independent. We know that exactly one entry of y is corrupted and differs from its correct value, but we don't know which sensor that is. A Julia program may split or append matrices and vectors and use `rank` to test consistency. Once enough accurate measurements have been identified, solving for x does not require an additional call to `rank`.

What are, respectively, (1) the minimum number of calls to `rank` needed to guarantee that the correct value of x can be recovered, and (2) the minimum number of calls to `rank` needed to guarantee that the corrupted measurement can be identified?

- ☐ 1 and 5
- ☐ 2 and 5
- ☐ 1 and 6
- ☐ 2 and 6