2.90. Matrices and signal flow graphs.

a) Find $A \in \mathbb{R}^{2 \times 2}$ such that $y = Ax$ in the system below:

\[
\begin{align*}
&x_1 
\quad \rightarrow \quad 2 
\quad \rightarrow \quad y_1 \\
&x_2 
\quad \rightarrow \quad 0.5 
\quad \rightarrow \quad + 
\quad \rightarrow \quad y_2
\end{align*}
\]

b) Find $B \in \mathbb{R}^{2 \times 2}$ such that $z = Bx$ in the system below:

\[
\begin{align*}
&x_1 
\quad \rightarrow \quad 2 
\quad \rightarrow \quad 0.5 
\quad \rightarrow \quad 2 
\quad \rightarrow \quad 0.5 
\quad \rightarrow \quad 2 
\quad \rightarrow \quad 0.5 
\quad \rightarrow \quad z_1 \\
&x_2 
\quad \rightarrow \quad 2 
\quad \rightarrow \quad 0.5 
\quad \rightarrow \quad 2 
\quad \rightarrow \quad 0.5 
\quad \rightarrow \quad 2 
\quad \rightarrow \quad 0.5 
\quad \rightarrow \quad z_2
\end{align*}
\]

Do this two ways: first, by expressing the matrix $B$ in terms of $A$ from the previous part (explaining why they are related as you claim); and second, by directly evaluating all possible paths from each $x_j$ to each $z_i$.

Solution.

a) By evaluating path gains we have

- Gain from $x_1$ to $y_1$. There is only one path with gain 2.
- Gain from $x_1$ to $y_2$. There is only one path with gain 0.5.
- Gain from $x_2$ to $y_1$. There are no paths and therefore the gain is 0.
- Gain from $x_2$ to $y_2$. There is only one path with gain 1.
and therefore
\[ A = \begin{bmatrix} 2 & 0 \\ 0.5 & 1 \end{bmatrix}. \]

b) Clearly \( B = A^4 \). Carrying out the multiplication gives
\[ B = \begin{bmatrix} 16 & 0 \\ 7.5 & 1 \end{bmatrix}. \]

Now by directly evaluating all possible path gains we get
- **Gain from \( x_1 \) to \( z_1 \).** There is only one path with gain \( 2 \times 2 \times 2 \times 2 = 16 \)
- **Gain from \( x_1 \) to \( z_2 \).** There are 4 possible paths. These paths have gains 0.5, \( 2 \times 0.5 \), \( 2 \times 2 \times 0.5 \) and \( 2 \times 2 \times 2 \times 0.5 \) that sum up to 7.5.
- **Gain from \( x_2 \) to \( z_1 \).** There are no paths and therefore the gain is 0.
- **Gain from \( x_2 \) to \( z_2 \).** There is only one path with gain 1.

and therefore we get the same \( B \) as expected.

\[ + + + + \]
\[ x_1 \]
\[ x_2 \]
\[ z_2 \]
\[ z_1 \]
\[ .5 .5 .5 .5 \]
\[ 2 2 2 2 \]
\[ + + + + \]

2.160. **Some matrices from signal processing.** We consider \( x \in \mathbb{R}^n \) as a signal, with \( x_i \) the (scalar) value of the signal at (discrete) time period \( i \), for \( i = 1, \ldots, n \). Below we describe several transformations of the signal \( x \), that produce a new signal \( y \) (whose dimension varies). For each one, find a matrix \( A \) for which \( y = Ax \).

a) **2× up-conversion with linear interpolation.** We take \( y \in \mathbb{R}^{2n-1} \). For \( i \) odd, \( y_i = x_{(i+1)/2} \). For \( i \) even, \( y_i = (x_{i/2} + x_{i/2+1})/2 \). Roughly speaking, this operation doubles the sample rate, inserting new samples in between the original ones using linear interpolation.

b) **2× down-sampling.** We assume here that \( n \) is even, and take \( y \in \mathbb{R}^{n/2} \), with \( y_i = x_{2i} \).

c) **2× down-sampling with averaging.** We assume here that \( n \) is even, and take \( y \in \mathbb{R}^{n/2} \), with \( y_i = (x_{2i-1} + x_{2i})/2 \).

**Solution.**
a)
\[
A_{\text{lin-int}} = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/2 & 1/2 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\]
2.200. Quadratic extrapolation of a time series. We are given a series \( z \) up to time \( t \). Using a quadratic model, we want to extrapolate, or predict, \( z(t + 1) \) based on the three previous elements of the series, \( z(t) \), \( z(t - 1) \), and \( z(t - 2) \). We’ll denote the predicted value of \( z(t + 1) \) by \( \hat{z}(t + 1) \). More precisely, you will find \( \hat{z}(t + 1) \) as follows.

a) Find the quadratic function \( f(\tau) = a_2\tau^2 + a_1\tau + a_0 \) which satisfies \( f(t) = z(t) \), \( f(t - 1) = z(t - 1) \), and \( f(t - 2) = z(t - 2) \). Then the extrapolated value is given by \( \hat{z}(t + 1) = f(t + 1) \). Show that

\[
\hat{z}(t + 1) = c \begin{bmatrix} z(t) \\ z(t - 1) \\ z(t - 2) \end{bmatrix},
\]

where \( c \in \mathbb{R}^{1 \times 3} \), and does not depend on \( t \). In other words, the quadratic extrapolator is a linear function. Find \( c \) explicitly.

b) Use the following Julia code to generate a time series \( z \):

```julia
   t = collect(1:1000);
   z = 5*\sin.(t/10 .+ 2) + 0.1 * \sin.(t) + 0.1*\sin.(2*t .- 5);
```

Use the quadratic extrapolation method from part (a) to find \( \hat{z}(t) \) for \( t = 4, \ldots, 1000 \). Find the relative root-mean-square (RMS) error, which is given by

\[
\left( \frac{1}{997} \sum_{j=4}^{1000} (\hat{z}(j) - z(j))^2 \right)^{1/2} / \left( \frac{1}{997} \sum_{j=4}^{1000} z(j)^2 \right)^{1/2}.
\]

Solution.

a) Setting \( f(t) = z(t) \), \( f(t - 1) = z(t - 1) \) and \( f(t - 2) = z(t - 2) \) gives the following system of linear equations:

\[
\begin{align*}
a_2t^2 + a_1t + a_0 &= z(t) \\
a_2(t - 1)^2 + a_1(t - 1) + a_0 &= z(t - 1) \\
a_2(t - 2)^2 + a_1(t - 2) + a_0 &= z(t - 2)
\end{align*}
\]
with solution
\[
\begin{align*}
a_0 &= (0.5t^2 - 1.5t + 1)z(t) + (2t - t^2)z(t - 1) + (0.5t^2 - 0.5t)z(t - 2) \\
a_1 &= (1.5 - t)z(t) + (2t - 2)z(t - 1) + (0.5 - t)z(t - 2) \\
a_2 &= 0.5z(t) - z(t - 1) + 0.5z(t - 2).
\end{align*}
\]
Substituting in \( \hat{z}(t + 1) = a_2(t + 1)^2 + a_1(t + 1) + a_0 \) and simplifying, we get
\[
\hat{z}(t + 1) = 3z(t) - 3z(t - 1) + z(t - 2).
\]
Hence,
\[
c = \begin{bmatrix} 3 & -3 & 1 \end{bmatrix}.
\]
Observe that \( c \) does not depend on \( t \), but the coefficients \( a_0, a_1 \) and \( a_2 \) do. In other words, the quadratic extrapolator \( f \) varies between samples, but its value at \( t + 1 \) is always given by the same combination of \( z(t), z(t - 1) \) and \( z(t - 2) \).

b) The relative RMS error is 0.097. In order to get an idea of how good the approximation is, we plot the first 100 samples:

![Graph of \( z(t) \) and \( \hat{z}(t) \)]

\[3.260. \text{Halfspace.} \text{ Suppose } a, b \in \mathbb{R}^n \text{ are two given points. Show that the set of points in } \mathbb{R}^n \text{ that are closer to } a \text{ than } b \text{ is a halfspace, i.e.}:
\[
\{ x \mid \|x - a\| \leq \|x - b\| \} = \{ x \mid c^T x \leq d \}
\]
for appropriate \( c \in \mathbb{R}^n \) and \( d \in \mathbb{R} \). Give \( c \) and \( d \) explicitly, and draw a picture showing \( a, b, \) and the halfspace.

**Solution.** It is easy to see geometrically what is going on: the hyperplane that goes right between \( a \) and \( b \) splits \( \mathbb{R}^n \) into two parts; the points closer to \( a \) (than \( b \)) and the points closer to \( b \) (than \( a \)). More precisely, the hyperplane is normal to the line through \( a \) and \( b \), and intersects that line at the midpoint between \( a \) and \( b \). Now that we have the idea, let’s try
to derive it algebraically. Let \( x \) belong to the set of points in \( \mathbb{R}^n \) that are closer to \( a \) than \( b \). Therefore \( \|x - a\| < \|x - b\| \) or \( \|x - a\|^2 < \|x - b\|^2 \) so

\[
(x - a)^T(x - a) < (x - b)^T(x - b).
\]

Expanding the inner products gives

\[
x^T x - x^T a - a^T x + a^T a < x^T x - x^T b - b^T x + b^T b
\]

or

\[
-2a^T x + a^T a < -2b^T x + b^T b
\]

and finally

\[
(b - a)^T x < \frac{1}{2}(b^T b - a^T a).
\]

Thus (1) is in the form \( c^T x < d \) with \( c = b - a \) and \( d = \frac{1}{2}(b^T b - a^T a) \) and therefore we have shown that the set of points in \( \mathbb{R}^n \) that are closer to \( a \) than \( b \) is a halfspace. Note that the hyperplane \( c^T x = d \) is perpendicular to \( c = b - a \).

3.350. Right inverses. This problem concerns the specific matrix

\[
A = \begin{bmatrix}
-1 & 0 & 0 & -1 & 1 \\
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0
\end{bmatrix}.
\]
This matrix is full rank (i.e., its rank is 3), so there exists at least one right inverse. In fact, there are many right inverses of $A$, which opens the possibility that we can seek right inverses that in addition have other properties. For each of the cases below, either find a specific matrix $B \in \mathbb{R}^{5 \times 3}$ that satisfies $AB = I$ and the given property, or explain why there is no such $B$. In cases where there is a right inverse $B$ with the required property, you must briefly explain how you found your $B$. You must also attach a printout of your Julia script or Jupyter notebook showing the verification that $AB = I$. (We’ll be very angry if we have to type in your $5 \times 3$ matrix into Julia to check it.) When there is no right inverse with the given property, briefly explain why there is no such $B$.

a) The second row of $B$ is zero.

b) The nullspace of $B$ has dimension one.

c) The third column of $B$ is zero.

d) The second and third rows of $B$ are the same.

e) $B$ is upper triangular, i.e., $B_{ij} = 0$ for $i > j$.

f) $B$ is lower triangular, i.e., $B_{ij} = 0$ for $i < j$.

Solution.

a) The second row of $B$ is zero. This means that the second column of $A$ isn’t used in forming $AB$. Let $\tilde{A}$ be the matrix $A$ with its second column removed, and let $\tilde{B}$ denote the matrix $B$ with its second row (which is supposed to be zero) removed. We have $\tilde{A} \tilde{B} = AB = I$, so $\tilde{B}$ is a right inverse of $\tilde{A}$. There is such a matrix if and only if $\tilde{A}$ is full rank, which it is. We can take $\tilde{B} = \tilde{A}^\top (\tilde{A} \tilde{A}^\top)^{-1}$. Finally to construct $B$ we simply insert a zero second row, moving rows 2, 3, 4 down by one. This gives the matrix

$$B = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}.$$  

There are other possible choices as well.

b) The nullspace of $B$ has dimension one. This means that $B$ has rank 2, so the rank of $AB$ is at most 2, which rules out the possibility that $AB = I$. So this is impossible.

c) The third column of $B$ is zero. This implies $B$ has a nullspace with dimension at least one, so by part (b) above, this is impossible too.

d) The second and third rows of $B$ are the same. Let $\tilde{B}$ denote $B$ with one of the (identical) rows 2 and 3 deleted. Then we have $AB = \tilde{A} \tilde{B}$, where $\tilde{A}$ is obtained from the matrix $A$ by replacing its second column with the sum of its second and third columns, and deleting its third column. Thus, we need to find a right inverse for $\tilde{A}$, provided it is full.
rank. It is, so we can take \( \tilde{B} = \tilde{A}^T(\tilde{A}A^T)^{-1} \). Finally to construct \( B \) we simply insert a second copy of the second row of \( \tilde{B} \) as a new third row. This gives

\[
B = \begin{bmatrix}
0 & 0 & 1/2 \\
0 & 1/2 & 0 \\
0 & 1/2 & 0 \\
1 & 0 & 1
\end{bmatrix}.
\]

This matrix also happens to be the pseudo-inverse of \( A \), \( B = A^T(AA^T)^{-1} \), and some of you noticed this immediately and used the pseudo-inverse to answer this question. That’s a fine answer; it was our mistake to choose \( A \) so that the pseudo-inverse satisfied this condition. In general, of course, it would not.

e) \( B \) is upper triangular, \textit{i.e.}, \( B_{ij} = 0 \) for \( i > j \). If \( B \) is upper triangular, then it has the form

\[
\begin{bmatrix}
\tilde{B} \\
0
\end{bmatrix},
\]

where \( \tilde{B} \) is square and upper triangular. If \( AB = I \), then \( \tilde{A} \tilde{B} = I \), where \( \tilde{A} \) is the matrix formed from the first 3 columns of \( A \). Thus we have \( \tilde{A} = \tilde{B}^{-1} \). But the inverse of an upper triangular matrix is also upper triangular, so unless \( \tilde{A} \) is upper triangular (and it isn’t, in this case), we can’t possibly have \( \tilde{A} \tilde{B} = I \). So there is no such \( B \) in this case.

f) \( B \) is lower triangular, \textit{i.e.}, \( B_{ij} = 0 \) for \( i < j \). Let’s label the columns of \( B \) as

\[
b_1, \quad b_2 = \begin{bmatrix} 0 \\ \tilde{b}_2 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 0 \\ 0 \\ \tilde{b}_3 \end{bmatrix},
\]

where \( \tilde{b}_2 \in \mathbb{R}^4 \) and \( \tilde{b}_3 \in \mathbb{R}^3 \). To say that \( AB = I \) is the same as saying that \( Ab_1 = e_1 \), \( Ab_2 = e_2 \), and \( Ab_3 = e_3 \), where \( e_1, e_2, e_3 \) are the unit vectors. We can solve these equations separately. The first equation is easy; the second we reduce to \( \tilde{A} \tilde{b}_2 = e_2 \), where here \( \tilde{A} \) is \( A \) with its first column removed. The third is handled similarly. These equations do have a solution; we get

\[
B = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1/2 & 0 \\
0 & 1/2 & 0 \\
0 & 0 & 1 \\
1 & 0 & 1
\end{bmatrix}.
\]

Another way: we set it up as a set of 9 linear equations (one for each entry of \( AB = I \)) in \( 5 + 4 + 3 = 12 \) variables. The variables are the first column of \( B \) (with 5 entries), the nonzero part of the second column of \( B \) (with 4 entries), and the nonzero part of the third second column of \( B \) (with 3 entries). We then attempt to solve these 9 equations in 12 variables. Some equations immediately give us the \( B \) matrix coefficients, while the
others can be solved by inspection to obtain a rather simple matrix

\[
B = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 1
\end{bmatrix}.
\]