2.100. A mass subject to applied forces. Consider a unit mass subject to a time-varying force \( f(t) \) for \( 0 \leq t \leq n \). Let the initial position and velocity of the mass both be zero. Suppose that the force has the form \( f(t) = x_j \) for \( j - 1 \leq t < j \) and \( j = 1, \ldots, n \). Let \( y_1 \) and \( y_2 \) denote, respectively, the position and velocity of the mass at time \( t = n \).

(a) Find the matrix \( A \in \mathbb{R}^{2 \times n} \) such that \( y = Ax \).

(b) For \( n = 4 \), find a sequence of input forces \( x_1, \ldots, x_n \) that moves the mass to position 1 with velocity 0 at time \( n \).

Solution. Let \( p(t) \) and \( v(t) \) denote, respectively, the position and velocity of the mass at time \( t \).

(a) The velocity is the integral of the applied force:

\[
v(t) = v(0) + \int_0^t f(\tau) \, d\tau \\
= v(0) + \sum_{j=1}^{[t]} \int_{j-1}^j f(\tau) \, d\tau + \int_{[t]}^t f(\tau) \, d\tau \\
= v(0) + \sum_{j=1}^{[t]} x_j \, d\tau + \int_{[t]}^t x_{[t]+1} \, d\tau \\
= v(0) + \sum_{j=1}^{[t]} (\tau x_j)_{\tau=j-1} + (\tau x_{[t]+1})_{\tau=[t]} \\
= v(0) + \sum_{j=1}^{[t]} x_j + (t - [t])x_{[t]+1}.
\]

In particular, because the mass is initially at rest (that is, \( v(0) = 0 \)), the final velocity is

\[
y_2 = v(n) = \sum_{j=1}^{n} x_j.
\]
Similarly, the position is the integral of the velocity:

\[
p(t) = p(0) + \int_0^t v(\tau) \, d\tau
\]

\[
= p(0) + \int_0^t (v(0) + (v(\tau) - v(0))) \, d\tau
\]

\[
= p(0) + v(0)t + \int_0^t (v(\tau) - v(0)) \, d\tau
\]

\[
= p(0) + v(0)t + \sum_{j=1}^{\lfloor \frac{t}{\tau} \rfloor} \int_{j-1}^j (v(\tau) - v(0)) \, d\tau + \int_{\lfloor \frac{t}{\tau} \rfloor}^t (v(\tau) - v(0)) \, d\tau
\]

\[
= p(0) + v(0)t + \sum_{j=1}^{\lfloor \frac{t}{\tau} \rfloor} \int_{j-1}^j \left( \sum_{k=1}^{\lfloor \tau \rfloor} x_k + (\tau - \lfloor \tau \rfloor) x_{\lfloor \tau \rfloor + 1} \right) \, d\tau
\]

\[
+ \int_{\lfloor \frac{t}{\tau} \rfloor}^t \left( \sum_{k=1}^{\lfloor \frac{\tau}{\tau} \rfloor} x_k + (\tau - \lfloor \tau \rfloor) x_{\lfloor \tau \rfloor + 1} \right) \, d\tau
\]

\[
= p(0) + v(0)t + \sum_{j=1}^{\lfloor \frac{t}{\tau} \rfloor} \left( \sum_{k=1}^{j-1} x_k + (\tau - (j-1)) x_j \right) \, d\tau
\]

\[
+ \int_{\lfloor \frac{t}{\tau} \rfloor}^t \left( \sum_{k=1}^{\lfloor \frac{\tau}{\tau} \rfloor} x_k + (\tau - \lfloor \tau \rfloor) x_{\lfloor \tau \rfloor + 1} \right) \, d\tau
\]

\[
= p(0) + v(0)t + \sum_{j=1}^{\lfloor \frac{t}{\tau} \rfloor} \left( \sum_{k=1}^{j-1} x_k + \frac{1}{2}(\tau - (j-1))^2 x_j \right)
\]

\[
+ \left( \sum_{k=1}^{\lfloor \frac{\tau}{\tau} \rfloor} \tau x_k + \frac{1}{2}(\tau - \lfloor \tau \rfloor)^2 x_{\lfloor \tau \rfloor + 1} \right)
\]

\[
= p(0) + v(0)t + \sum_{j=1}^{\lfloor \frac{t}{\tau} \rfloor} \left( \sum_{k=1}^{j-1} x_k + \frac{1}{2} x_j \right) + \sum_{k=1}^{\lfloor \frac{\tau}{\tau} \rfloor} \left( x_k + \frac{1}{2}(t - \lfloor \tau \rfloor)^2 x_{\lfloor \tau \rfloor + 1} \right)
\]

\[
= p(0) + v(0)t + \sum_{j=1}^{\lfloor \frac{t}{\tau} \rfloor} \sum_{k=1}^{j-1} x_k + \sum_{j=1}^{\lfloor \frac{\tau}{\tau} \rfloor} \frac{1}{2} x_j + \sum_{k=1}^{\lfloor \frac{t}{\tau} \rfloor} x_k + \frac{1}{2}(t - \lfloor \tau \rfloor)^2 x_{\lfloor \tau \rfloor + 1}
\]

\[
= p(0) + v(0)t + \sum_{k=1}^{\lfloor \frac{t}{\tau} \rfloor} \sum_{j=1}^{\lfloor \tau \rfloor} x_k + \sum_{k=1}^{\lfloor \frac{t}{\tau} \rfloor} \frac{1}{2} x_k + \sum_{k=1}^{\lfloor \frac{\tau}{\tau} \rfloor} x_k + \frac{1}{2}(t - \lfloor \tau \rfloor)^2 x_{\lfloor \tau \rfloor + 1}
\]

\[
= p(0) + [t] k \right) x_k + \sum_{k=1}^{\lfloor \frac{t}{\tau} \rfloor} x_k + \frac{1}{2}(t - \lfloor \tau \rfloor)^2 x_{\lfloor \tau \rfloor + 1}
\]

\[
= p(0) + v(0)t + \sum_{k=1}^{\lfloor \frac{t}{\tau} \rfloor} \left( \frac{1}{2} + (t - \lfloor \tau \rfloor) \right) x_k + \frac{1}{2}(t - \lfloor \tau \rfloor)^2 x_{\lfloor \tau \rfloor + 1}
\]
\[ = p(0) + v(0)t + \sum_{k=1}^{\lfloor t \rfloor} \left( t - k + \frac{1}{2} \right) x_k + \frac{1}{2} (t - \lfloor t \rfloor)^2 x_{\lfloor t \rfloor + 1}. \]

In particular, because the mass is initially at rest at the origin (that is, \( p(0) = 0 \) and \( v(0) = 0 \)), the final position is
\[ y_1 = p(n) = \sum_{j=1}^{n} (n - k + \frac{1}{2}) x_j. \]

Thus, we obtain the following system of linear equations:
\[ y_1 = \sum_{j=1}^{n} (n - j + \frac{1}{2}) x_j, \]
\[ y_2 = \sum_{j=1}^{n} x_j. \]

Since \( A_{ij} \) gives the coefficient of \( x_j \) in our expression for \( y_i \), we have that
\[ A_{1j} = n - j + \frac{1}{2} \quad \text{and} \quad A_{2j} = 1, \quad j = 1, \ldots, n. \]

More concretely, we have that
\[ A = \begin{bmatrix} n - \frac{1}{2} & n - \frac{3}{2} & \cdots & \frac{3}{2} & \frac{1}{2} \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}. \]

b) We want to solve the following system of linear equations:
\[
\begin{bmatrix}
\frac{7}{2} & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\
1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
=
\begin{bmatrix}
1 \\
0
\end{bmatrix}.
\]

This system is underdetermined, and has infinitely many solutions. Suppose we choose \( x_2 = x_3 = 0 \). Then, we are left with the system
\[
\begin{bmatrix}
\frac{7}{2} & \frac{1}{2} \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_4
\end{bmatrix}
=
\begin{bmatrix}
1 \\
0
\end{bmatrix}.
\]

The second equation implies that \( x_4 = -x_1 \). Then, the first equation becomes
\[
\frac{7}{2} x_1 + \frac{1}{2} x_4 = \frac{7}{2} x_1 - \frac{1}{2} x_1 = 3x_1 = 1.
\]

Solving this equation, we find that \( x_1 = \frac{1}{3} \). Substituting this value into our expression for \( x_4 \) gives \( x_4 = -x_1 = -\frac{1}{3} \). Thus, one sequence of input forces that moves the mass to position 1 with velocity 0 at time \( n \) is
\[ x = \begin{bmatrix}
\frac{1}{3} \\
0 \\
0 \\
-\frac{1}{3}
\end{bmatrix}. \]
2.120. Counting sequences in a language or code. We consider a language or code with an alphabet of \( n \) symbols \( 1, 2, \ldots, n \). A sentence is a finite sequence of symbols, \( k_1, \ldots, k_m \) where \( k_i \in \{1, \ldots, n\} \). A language or code consists of a set of sequences, which we will call the allowable sequences. A language is called Markov if the allowed sequences can be described by giving the allowable transitions between consecutive symbols. For each symbol we give a set of symbols which are allowed to follow the symbol. As a simple example, consider a Markov language with three symbols \( 1, 2, 3 \). Symbol 1 can be followed by 1 or 3; symbol 2 must be followed by 3; and symbol 3 can be followed by 1 or 2. The sentence 1132313 is allowable (i.e., in the language); the sentence 1132312 is not allowable (i.e., not in the language). To describe the allowed symbol transitions we can define a matrix \( A \in \mathbb{R}^{n \times n} \) by

\[
A_{ij} = \begin{cases} 
1 & \text{if symbol } i \text{ is allowed to follow symbol } j \\
0 & \text{if symbol } i \text{ is not allowed to follow symbol } j.
\end{cases}
\]

a) Let \( B = A^r \). Give an interpretation of \( B_{ij} \) in terms of the language.

b) Consider the Markov language with five symbols \( 1, 2, 3, 4, 5 \), and the following transition rules:

- 1 must be followed by 2 or 3
- 2 must be followed by 2 or 5
- 3 must be followed by 1
- 4 must be followed by 4 or 2 or 5
- 5 must be followed by 1 or 3

Find the total number of allowed sentences of length 10. Compare this number to the simple code that consists of all sequences from the alphabet (i.e., all symbol transitions are allowed). In addition to giving the answer, you must explain how you solve the problem. Do not hesitate to use Julia.

Solution.

a) If \( B = A^k \), then \( B_{ij} \) is the number of sequences of length \( k + 1 \) that start with symbol \( j \) and end with symbol \( i \).

Here is a formal proof. Let \( S_L(i, j) \) denote the set of sentences of length \( L \) that start with symbol \( j \) and end with symbol \( i \):

\[
S_L(i, j) = \{(k_1 = j, k_2, \ldots, k_{L-1}, k_L = i) \in \mathbb{N}_n^L | A_{k_{l+1}k_{l+1}} = 1 \text{ for all } \ell = 1, \ldots, L - 1\}.
\]

We claim that

\[
(A^p)_{ij} = |S_{p+1}(i, j)|
\]

for all \( i, j \in \mathbb{N}_n \) and \( p \in \mathbb{N} \). First, we introduce some notation. Given a finite sequence of symbols \( (k_1, \ldots, k_L) \) and a symbol \( k_{L+1} \), we define

\[
(k_1, \ldots, k_L) + k_{L+1} = (k_1, \ldots, k_L, k_{L+1}).
\]
Thus, $+$ denotes the operation of appending a symbol to a finite sequence of symbols. Similarly, given a set $S$ of finite sequences of symbols, we define

$$S + j = \{ s + j \mid s \in S \}.$$  

Finally, we define

$$\phi(i) = \{ j \in \mathbb{N}_n \mid \text{symbol } i \text{ is allowed to follow symbol } j \} = \{ j \in \mathbb{N}_n \mid A_{ij} = 1 \}.$$  

First, we prove the claim when $p = 1$:

$$(A^1)_{ij} = |S_2(i, j)|.$$  

Note that $|S_2(i, j)| = 1$ if $(j, i)$ is a sentence in the language (that is, symbol $i$ is allowed to follow symbol $j$; or, equivalently, $A_{ij} = 1$), and $|S_2(i, j)| = 0$ otherwise (that is, symbol $i$ is not allowed to follow symbol $j$; or, equivalently, $a_{ij} = 0$). In either case, we have that $|S_2(i, j)| = A_{ij} = (A^1)_{ij}$. Now suppose that $(A^p)_{ij} = |S_{p+1}(i, j)|$ for some $p \in \mathbb{N}$. We can partition $S_{p+2}(i, j)$ based on the penultimate symbol:

$$S_{p+2}(i, j) = \bigsqcup_{k \in \phi(i)} (S_{p+1}(k, j) + i).$$  

Since the size of a disjoint union is equal to the sum of the sizes of the sets forming the union, we have that

$$|S_{p+2}(i, j)| = \sum_{k \in \phi(i)} |S_{p+1}(k, j) + i| = \sum_{k \in \phi(i)} |S_{p+1}(k, j)|.$$  

Because $A_{ik} = 1$ for $k \in \phi(i)$, and $A_{ik} = 0$ for $k \notin \phi(i)$, we have that

$$|S_{p+2}(i, j)| = \sum_{k=1}^{n} A_{ik} |S_{p+1}(k, j)|.$$  

Using the induction hypothesis, we have that $|S_{p+1}(k, j)| = (A^p)_{kj}$, and hence that

$$|S_{p+2}(i, j)| = \sum_{k=1}^{n} A_{ik} (A^p)_{kj} = (A A^p)_{ij} = (A^{p+1})_{ij}.$$  

By induction, this proves that our interpretation of $(A^p)_{ij}$ is correct for all $p \in \mathbb{N}$.

b) For the given Markov language we can find the number of allowed sequences of length 10 by simply adding all the entries of the matrix $A^9$. Form the description of the rules we have

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$  

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Using the Julia command $B = A^9$ we find

\[
B = A^9 = \begin{bmatrix}
41 & 49 & 24 & 113 & 37 \\
55 & 65 & 31 & 150 & 49 \\
42 & 49 & 23 & 113 & 37 \\
0 & 0 & 0 & 1 & 0 \\
31 & 37 & 18 & 86 & 28
\end{bmatrix}.
\]

(Note that there is only one word that ends with 4 (i.e., 4444444444, because 4 can only follow 4.) Adding all the elements of $B$, using for example the Julia command \texttt{sum(B)}, we find that the total number of allowed sequences of length 10 is 1079. Finally, we can compare this number to the simple code that consists of all sequences from the alphabet. Of course there are $5^{10} = 9765625$ such sequences. Just to check our method, we can also compute this number the same way as above, by forming the matrix

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix},
\]

(which means any symbol can follow any symbol), then find $B = A^9$, and adding all the entries of the matrix $B$. (Yes, you do get the same number as above . . .)

2.150. Gradient of some common functions. Recall that the gradient of a differentiable function $f: \mathbb{R}^n \to \mathbb{R}$, at a point $x \in \mathbb{R}^n$, is defined as the vector

\[
\nabla f(x) = \left[\begin{array}{c}
\frac{\partial f}{\partial x_1} \\
\vdots \\
\frac{\partial f}{\partial x_n}
\end{array}\right],
\]

where the partial derivatives are evaluated at the point $x$. The first order Taylor approximation of $f$, near $x$, is given by

\[
\hat{f}_{\text{tay}}(z) = f(x) + \nabla f(x)^T (z - x).
\]

This function is affine, i.e., a linear function plus a constant. For $z$ near $x$, the Taylor approximation $\hat{f}_{\text{tay}}$ is very near $f$. Find the gradient of the following functions. Express the gradients using matrix notation.

a) $f(x) = a^T x + b$, where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$.

b) $f(x) = x^T Ax$, for $A \in \mathbb{R}^{n \times n}$.

c) $f(x) = x^T Ax$, where $A = A^\top \in \mathbb{R}^{n \times n}$. (Yes, this is a special case of the previous one.)
Solution.

a) Since \( f(x) = a_1x_1 + \cdots + a_nx_n + b \), we have \( \frac{\partial f}{\partial x_i} = a_i \), so \( \nabla f(x) = a \). So the gradient of an affine function is constant, and equal to the vector associated with the linear part of the function.

b) We write out \( f \) explicitly as
\[
f(x) = \sum_{i=1}^{n} x_i (Ax)_i = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j.
\]
Therefore we have
\[
\frac{\partial f}{\partial x_k} = \sum_{i=1}^{n} A_{ik} x_i + \sum_{j=1}^{n} A_{kj} x_j = (A^T x)_k + (Ax)_k.
\]
So \( \nabla f(x) = (A + A^T)x \).

c) By the previous problem, we have \( \nabla f(x) = 2Ax \).

2.180. Paths and cycles in a directed graph. We consider a directed graph with \( n \) nodes. The graph is specified by its node adjacency matrix \( A \in \mathbb{R}^{n \times n} \), defined as
\[
A_{ij} = \begin{cases} 
1 & \text{if there is an edge from node } j \text{ to node } i \\
0 & \text{otherwise.}
\end{cases}
\]
Note that the edges are oriented, i.e., \( A_{34} = 1 \) means there is an edge from node 4 to node 3. For simplicity we do not allow self-loops, i.e., \( A_{ii} = 0 \) for all \( i, 1 \leq i \leq n \). A simple example illustrating this notation is shown below.

![Diagram of a directed graph](image)

The node adjacency matrix for this example is
\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}.
\]

In this example, nodes 2 and 3 are connected in both directions, i.e., there is an edge from 2 to 3 and also an edge from 3 to 2. A path of length \( l > 0 \) from node \( j \) to node \( i \) is a sequence \( s_0 = j, s_1, \ldots, s_l = i \) of nodes, with \( A_{s_{k+1},s_k} = 1 \) for \( k = 0, 1, \ldots, l - 1 \). For example, in the
graph shown above, 1, 2, 3, 2 is a path of length 3. A cycle of length \( l \) is a path of length \( l \), with the same starting and ending node, with no repeated nodes other than the endpoints. In other words, a cycle is a sequence of nodes of the form \( s_0, s_1, \ldots, s_{l-1}, s_0 \), with
\[
A_{s_1,s_0} = 1, \quad A_{s_2,s_1} = 1, \quad \ldots \quad A_{s_{l-1},s_{l-2}} = 1,
\]
and
\[
s_i \neq s_j \text{ for } i \neq j, \quad i, j = 0, \ldots, l-1.
\]
For example, in the graph shown above, 1, 2, 3, 4, 1 is a cycle of length 4. The rest of this problem concerns a specific graph, given in the file `directed_graph.json` on the course website. For each of the following questions, you must give the answer explicitly (for example, enclosed in a box). You must also explain clearly how you arrived at your answer.

a) What is the length of a shortest cycle? (Shortest means minimum length.)

b) What is the length of a shortest path from node 13 to node 17? (If there are no paths from node 13 to node 17, you can give the answer as ‘infinity’.)

c) What is the length of a shortest path from node 13 to node 17, that does not pass through node 3?

d) What is the length of a shortest path from node 13 to node 17, that does pass through node 9?

e) Among all paths of length 10 that start at node 5, find the most common ending node.

f) Among all paths of length 10 that end at node 8, find the most common starting node.

g) Among all paths of length 10, find the most common pair of starting and ending nodes. In other words, find \( i, j \) which maximize the number of paths of length 10 from \( i \) to \( j \).

**Solution.**

a) Recall that \( (A^k)_{ij} \) gives the number of paths of length \( k \) from node \( j \) to node \( i \). Thus, \( (A^k)_{ii} \) is the number of paths of length \( k \) from node \( i \) to itself. Now imagine increasing \( k \) from \( k = 1 \) to \( k = 2, k = 3, \) and so on. We find the smallest \( k \) for which \( (A^k)_{ii} > 0 \). This \( k \) is the length of the smallest path from \( i \) to itself. This path is in fact also a cycle, since it cannot repeat nodes. (If it repeated nodes, there would have been a shorter path from \( i \) to itself.) Now let’s solve the problem. To find the length of a shortest cycle, find the smallest \( k \) such that \( (A^k)_{ii} > 0 \) for some \( i \). Note \( k \leq n \), because if a cycle exists then it is at most of length \( n \), where \( n \) is the number of nodes in the graph. The smallest cycle is of length 6.

b) To find the length of a shortest path from node 13 to node 17, find the smallest \( k \) such that \( (A^k)_{17,13} > 0 \). The shortest path from node 13 to node 17 is of length 4.

c) To find the shortest path from node 13 to node 17, that does not pass though node 3, remove node 3 from the graph and then find the shortest path from node 13 to node 17. The new adjacency matrix \( B \) for the graph is obtained by removing the 3rd row and column of the matrix \( A \). Then find the smallest \( k \) such that \( (B^k)_{17,13} > 0 \). The shortest path from node 13 to node 17, that does not pass through node 3 is of length 5.
d) To find the smallest path from node 13 to node 17, that does pass through node 9, find the shortest path from node 13 to node 9 and the shortest path from node 9 to node 17. The shortest path from node 13 to node 17, that does pass through node 9 is of length 10.

e) The matrix $A^{10}$ gives the number of paths of length 10, i.e., $(A^{10})_{ij}$ is the number of paths of length 10 that start at node $j$ and end at node $i$. The 5th column of the matrix $A^{10}$ gives the number of paths of length 10 that start at node 5 and end at nodes 1, 2, . . . , 20 respectively. The index of the maximum entry of this column gives the most common ending node for paths of length 10 starting at node 5. The most common ending node for paths of length 10 starting at node 5, is 5.

f) The 8th row of the matrix $A^{10}$ gives the number of paths that end at node 8. The index of the maximum entry of this row gives the most common starting node for paths of length 10 ending at node 8. The most common starting node for paths of length 10 ending at node 8, is 8.

g) The most common source/destination pair for paths of length 10 is the index of the maximum entry of $A^{10}$, i.e., if $(i, j)$ is the most common source/destination pair then no other number in the matrix $A^{10}$ is greater than $(A^{10})_{ji}$. The most common source/destination pair for paths of length 10 is $(8, 17)$.

3.240. Price elasticity of demand. The demand for $n$ different goods is a function of their prices:

$$q = f(p),$$

where $p$ is the price vector, $q$ is the demand vector, and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the demand function. The current price and demand are denoted $p^*$ and $q^*$, respectively. Now suppose there is a small price change $\delta p$, so $p = p^* + \delta p$. This induces a change in demand, to $q \approx q^* + \delta q$, where

$$\delta q \approx Df(p^*)\delta p,$$

where $Df$ is the derivative or Jacobian of $f$, with entries

$$Df(p^*)_{ij} = \frac{\partial f_i}{\partial p_j}(p^*).$$

This is usually rewritten in term of the elasticity matrix $E$, with entries

$$E_{ij} = \frac{\partial f_i}{\partial p_j}(p^*) \frac{1}{q_i^*} \frac{1}{p_j^*},$$

so $E_{ij}$ gives the relative change in demand for good $i$ per relative change in price $j$. Defining the vector $y$ of relative demand changes, and the vector $x$ of relative price changes,

$$y_i = \frac{\delta q_i}{q_i^*}, \quad x_j = \frac{\delta p_j}{p_j^*},$$

we have the linear model $y = Ex$.

Here are the questions:

a) What is a reasonable assumption about the diagonal elements $E_{ii}$ of the elasticity matrix?
b) Goods \( i \) and \( j \) are called \textit{substitutes} if they provide a similar service or other satisfaction (\textit{e.g.}, train tickets and bus tickets, cake and pie, etc.). They are called \textit{complements} if they tend to be used together (\textit{e.g.}, automobiles and gasoline, left and right shoes, etc.). For each of these two generic situations, what can you say about \( E_{ij} \) and \( E_{ji} \)?

c) Suppose the price elasticity of demand matrix for two goods is

\[
E = \begin{bmatrix}
-1 & -1 \\
-1 & -1 \\
\end{bmatrix}.
\]

Describe the nullspace of \( E \), and give an interpretation (in one or two sentences). What kind of goods could have such an elasticity matrix?

\textbf{Solution.}

a) The \( i \)th diagonal entry of \( E \) relates \( y_i \) to \( x_i \), \textit{i.e.}, the demand for the \( i \)th good to its price. When the price of a product is increased, and all other prices are held constant, the demand for that product can be expected to decrease. Hence, the diagonal elements of \( E \) should be negative. (Whether any good with a positive elasticity exists at all is a debated question, but most economists’ answer is no.)

b) The entry \( E_{ij} \) relates the demand for good \( i \) to the price of good \( j \). A price increase in good \( j \) leads to a decreased demand for that good. If good \( i \) is a \textit{substitute}, it also leads to an \textit{increased} demand for good \( i \), as some of the consumption switches to what now seems a more attractive price. Hence, \( E_{ij} \) is positive. The same argument tells us that \( E_{ji} \) is positive when goods \( i \) and \( j \) are substitutes.

If the goods are \textit{complements}, the converse is true. Since the consumption of good \( i \) is associated with consumption of good \( j \), a \textit{decreased} demand for good \( i \) follows from the decreased demand for good \( j \). Hence, \( E_{ij} \) (and \( E_{ji} \)) is negative.

c) The nullspace consists of the vectors of the form

\[
x = \alpha \begin{bmatrix}
1 \\
-1 \\
\end{bmatrix},
\]

with the scaling factor \( \alpha \) any real number. If the two prices are changed by an equal amount, with one being increased and the other decreased, the demand is not affected at all. This can happen if the two goods are \textit{perfect complements}, that is, if they are always consumed in fixed proportions (such as right and left shoes!) In this case, consumers only care about the total cost of the two goods, not the individual prices.