EE263: Introduction to Linear Dynamical Systems
Review Session 1

Outline

• administrative information

• examples
Administrative information

- website: http://ee263.stanford.edu/
- forum: https://piazza.com/stanford/autumn2014/ee263
- email: ee263-aut1415-staff@lists.stanford.edu
Teaching assistants

- Aditya Timmaraju
- Reza Takapoui
- Bobbie Chern

TA office hours:

- Wednesdays  10am - 1pm  Packard 104 (Aditya)
- Thursdays    1pm - 4pm   Packard 104 (Reza)
- Fridays      2pm - 5pm   Packard 106 (Bobbie)
Sections

• Fridays 9-9:50am, Nvidia Auditorium
• attendance optional
• all notes posted
• sessions are recorded
Homeworks

Policy

• due Fridays 5 pm (EE263 bin, 2nd floor Packard)

• no late hws accepted

• graded roughly on a course scale 1–10
Matlab basics


```matlab
>> a = 1
a =
    1

>> b = 1;

>> c = [1 2 3]
c =
    1  2  3

>> c+2
ans =
    3  4  5
```
Matlab basics (contd...)

```matlab
>> c
  c =
     1  2  3

>> d = [4;5;6]
  d =
    4
    5
    6

>> c*d
  ans =
    32

>> e = d.'
  e =
     4  5  6
```
Matlab basics (contd...)

>> c+e
ans =
 5    7    9

>> c.*e
ans =
   4   10   18

>> A = [1 2; 3 4]
A =
   1    2
   3    4

>> A*[1;1]
ans =
   3
   7
Matlab basics (contd...)

>> A(2,1)
ans =
 3

>> A(:,1)
ans =
 1
 3

>> A(2,:)
ans =
 3  4

>> t = 0:2:10
 t =
 0  2  4  6  8  10
Table as a matrix

- nutrition chart

<table>
<thead>
<tr>
<th>Nutrient</th>
<th>Vegetable</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x_1$ $x_2$  $\cdots$ $x_n$</td>
</tr>
<tr>
<td>$y_1$</td>
<td>0.50 0.75  $\cdots$ 0.9</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$ $\vdots$ $\vdots$ $\vdots$</td>
</tr>
<tr>
<td>$y_m$</td>
<td>2.05 0.01  $\cdots$ 0.45</td>
</tr>
</tbody>
</table>

- vector $x \in \mathbb{R}^n$ is the vegetable diet; $x_j$ is amount of vegetable $j$
- vector $y \in \mathbb{R}^m$ is the nutrients; $y_i$ is the amount of nutrient $i$
- $y = Ax$ gives the nutrients as a function of the vegetable diet
- $A_{ij} =$ amount of nutrient $i$ in 1 unit of vegetable $j$
Examples

• \( x \in \mathbb{R}^n \)

• find \( A \) for which \( y = Ax \) is the running average of \( x \), i.e.,

\[
y_i = \frac{1}{i} \sum_{j=1}^{i} x_j, \quad i = 1, \ldots, n
\]

Solution.

\[
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
1/2 & 1/2 & 0 & 0 & \cdots & 0 \\
1/3 & 1/3 & 1/3 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1/n & 1/n & 1/n & 1/n & \cdots & 1/n
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
\]
Examples

- Creating $A$ in Matlab

\begin{verbatim}
n=5;
A = zeros(n,n);
for i=1:n
    A(i,1:i) = 1/i;
end
A
\end{verbatim}

\[
A = \\
\begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 \\
0.5000 & 0.5000 & 0 & 0 & 0 \\
0.3333 & 0.3333 & 0.3333 & 0 & 0 \\
0.2500 & 0.2500 & 0.2500 & 0.2500 & 0 \\
0.2000 & 0.2000 & 0.2000 & 0.2000 & 0.2000 \\
\end{bmatrix}
\]
Examples (contd...)

• what is $y = Ax$, with $A = (1/n)11^T$?

Solution.

• $1^T x = x_1 + \cdots + x_n$, the sum of the $x_i$

• $(1/n)1^T x$ is the average of the $x_i$

• $y = (1/n)11^T x$ is a vector of size $n$, with each component equal to the average of the $x_i$
Linear mechanical system

\[ M\ddot{q} + D\dot{q} + Kq = f \]

- \( q(t) \in \mathbb{R}^k \) is the vector of deflections
- \( f(t) \in \mathbb{R}^k \) is the vector of externally applied forces
- \( M, D, K \in \mathbb{R}^{k \times k} \) are the mass, damping, and stiffness matrices, respectively
Linear system equations

- assume $M \in \mathbb{R}^{k \times k}$ full rank

- let state $x = (q, \dot{q}) = [q^T \dot{q}^T]^T$

- let action $u = f$

- let output $y = q$

- write $\dot{x} = Ax + Bu, \ y = Cx + Du$

solution

$$
\begin{bmatrix}
\dot{q} \\
\ddot{q}
\end{bmatrix} =
\begin{bmatrix}
0 \\
-M^{-1}K & -M^{-1}D
\end{bmatrix}
\begin{bmatrix}
q \\
\dot{q}
\end{bmatrix} +
\begin{bmatrix}
0 \\
M^{-1}
\end{bmatrix} f
$$

$$
q =
\begin{bmatrix}
I \\
0
\end{bmatrix}
\begin{bmatrix}
q \\
\dot{q}
\end{bmatrix}
$$
Polynomial evaluation

- polynomial \( p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1} \)

- points \( t_1, \ldots, t_m \in \mathbb{R} \)

- \( y_i = p(t_i) \), i.e., \( p \) evaluated at points \( t_1, \ldots, t_m \)

- can write as \( y = Ta \):
  \[
  \begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_m 
  \end{bmatrix}
  =
  \begin{bmatrix}
  1 & t_1 & \cdots & t_1^{n-1} \\
  1 & t_2 & \cdots & t_2^{n-1} \\
  \vdots & \vdots & \cdots & \vdots \\
  1 & t_m & \cdots & t_m^{n-1} 
  \end{bmatrix}
  \begin{bmatrix}
  a_0 \\
  a_1 \\
  \vdots \\
  a_{n-1}
  \end{bmatrix}
  \]

- so \( y \) depends linearly on \( a \), can do parameter estimation etc.
Node adjacency matrix

- graph with \( n \) nodes and (undirected) edges

- node adjacency matrix \( A \in \mathbb{R}^{n \times n} \) is given by
  \[
  A_{ij} = \begin{cases} 
  1 & \text{there is an edge connecting node } i \text{ and node } j \\
  0 & \text{otherwise}
  \end{cases}
  \]

- what is meaning of \( (i, j) \)th entry of \( A^k \)?
Node adjacency matrix (cont...) 

Solution.

• let’s start with $k = 2$

• $A_{ik}A_{kj} = 1$ only if there is an edge from node $i$ to node $k$, and an edge from node $k$ to node $j$, i.e., there is a path of length 2 between $i$ and $j$, passing through node $k$

• $(A^2)_{ij} = \sum_{k=1}^{n} A_{ik}A_{kj}$ which is the number of paths of length 2 between $i$ and $j$

• in general case, $(A^k)_{ij}$ is the number of paths of length $k$ between nodes $i$ and $j$
Node incidence matrix

- graph with \( n \) nodes and \( m \) directed edges

- node incidence matrix \( A \in \mathbb{R}^{n \times m} \) is defined as

\[
A_{ij} = \begin{cases} 
1 & \text{edge } j \text{ enters (points into) node } i \\
-1 & \text{edge } j \text{ leaves (points out of) node } i \\
0 & \text{otherwise.}
\end{cases}
\]

- what does \( y = Ax \) mean?

- what does \( z = A^T w \) mean?

- what is \( AA^T \)?
Node incidence matrix (contd...)

\textit{Solution.}

- defining $x_j$ to be flow rate on edge $j$, $y_i$ is the total flow into node $i$ (out of, if negative)

- for $w_j \in \{0, 1\}$, $z_i$ is the sum rate on edge $i$

- $(AA^T)_{ii} = \text{number of edges connected to the } i\text{th node}$

- for $i \neq j$, $(AA^T)_{ij} = -1$, only if there is an edge from node $i$ to node $j$