EE263 Final Exam, December 2023

- This is a 24-hour take-home exam. Please turn it in on Gradescope. Be aware that you *must* turn it in within 24 hours of downloading it. After that, Gradescope will not let you turn it in and we cannot accept it.
- You may use any books, notes, or computer programs. You may not discuss the exam or course material with others, or work in a group.
- The exam should not be discussed at all, with anyone, until 12/13 after everyone has taken it.
- If you have a question, please submit a private question on Ed, or email the staff mailing list. We have tried very hard to make the exam unambiguous and clear, so unless there is a mistake on the exam we're unlikely to say much.
- We expect your solutions to be legible, neat, and clear. Do not hand in your rough notes, and please try to simplify your solutions as much as you can. We will deduct points from solutions that are technically correct, but much more complicated than they need to be.
- Please check your email during the exam, just in case we need to send out a clarification or other announcement.
- Start each question on a new page, and make sure to correctly assign pages to problems in gradescope.
- When a problem involves some computation (say, using Julia), we do not want just the final answers. We want a clear discussion and justification of exactly what you did as well as the final numerical result.
- **You must turn in your code**. Include the code in your pdf submission.
- In the portion of your solutions where you explain the mathematical approach, you *cannot* refer to Julia operators, such as the backslash operator. (You can, of course, refer to inverses of matrices, or any other standard mathematical constructs.)
- Some of the problems require you to download data or other files. These files can be found at the URL

  `https://ee263.stanford.edu/grandfinale23`

- Good luck!

1. **Hovercraft racing.** You've just entered into the world's first Hovercraft Grand Prix! This high-stakes competition will put the controllability of your homemade hovercraft to the test. The primary goal of your hovercraft is to intercept **proximity targets** laid out in a track. The targets must be hit at particular times (so it's not really a race, it's a test of accuracy.) Your controller can apply forces in the x and y directions.

   More specifically, you can apply piecewise constant force inputs $u(t) = u_d(k)$ for $k \le t < k+1$ where $k \in \mathbb{Z}^+$ and $u(t) \in \mathbb{R}^2$. The state of the system is

$$x(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}$$

   where $q(t) \in \mathbb{R}^2$ is the position of the hovercraft at time t. The starting line of the race is at $(0,0)$ and the hovercraft must be stationary until the race begins. The hovercraft has mass 1.

   a) Find matrices A and B that model the hovercraft's dynamics exactly as a discrete-time linear dynamical system of the form

$$x(k+1) = Ax(k) + Bu_d(k)$$

   b) You are given the $m = 4$ target positions $p_1 = (1,1)$, $p_2 = (0,2)$, $p_3 = (-1,1)$ and $p_4 = (0,0)$, and target times $t_1 = 10$, $t_2 = 40$, $t_3 = 50$, and $t_4 = 60$. Let $t_{\max} = 65$. Find the sequence of forces $u_d(0), \ldots, u_d(t_{\max})$ that minimizes

$$\sum_{k=0}^{t_{\max}} \|u_d(k)\|^2$$

   such that the hovercraft passes through each of the target positions at the corresponding target time. Plot the trajectory of the hovercraft in $\mathbb{R}^2$. Plot the components of the input force vs. time.

   c) New regulations have been issued this year that now limit the total thrust. We would like to trade off

$$J_1 = \sum_{i=1}^{m} \|q(t_i) - p_i\|^2$$

   and

$$J_2 = \sum_{k=0}^{t_{\max}} \|u_d(k)\|^2$$

   While we may not be able to pass through each target exactly, we still get points for getting as close as possible to each one. Describe a method to do this. Plot the optimal trade-off curve of J_2 versus J_1.

   d) Find the controller inputs that minimize J_1 while achieving the total thrust limit

$$J_2 \le 0.005$$

   Plot the trajectory of the hovercraft in $\mathbb{R}^2$. Plot the components of the input force vs. time.

2. Heat flow. We consider here an example of a simple model for heat flow. We have a directed graph with m edges and n nodes. The incidence matrix $A \in \mathbb{R}^{m \times n}$ is defined by

$$A_{ij} = \begin{cases} 1 & \text{if edge } j \text{ points to node } i \\ -1 & \text{if edge } j \text{ starts at node } i \\ 0 & \text{otherwise} \end{cases}$$

An edge cannot start and end at the same node. The temperature at node i at time t is $x_i(t)$. Suppose we add heat to node i are rate $s_i(t)$. Let $f_k(t)$ be the amount of heat flowing through edge k at time t. Then we have

$$\dot{x}(t) = s(t) + Af(t)$$

Now the rate of heat flow on an edge is proportional to the temperature difference at its endpoints, so we have

$$Kf = -A^{\mathsf{T}}x$$

where $K \in \mathbb{R}$ is a constant. Then $1/K$ is the *thermal conductivity* of the edge. For convenience, in this question we will assume $K = 1$. This model is a linear dynamical system of the form

$$\dot{x}(t) = Qx(t) + s$$

where $Q = -AA^{\mathsf{T}}$.

a) Show that for any graph Q has at least one zero eigenvalue. What is the corresponding eigenvector?

b) We will first consider the unforced system

$$\dot{x}(t) = Qx(t)$$

What are the equilibirium solutions of this system? Give a brief one-sentence interpretation.

c) A matrix $P \in \mathbb{R}^{n \times n}$ is called *off-diagonal nonnegative* if

$$P_{ij} \geq 0 \quad \text{for } i \neq j$$

Show that Q is off-diagonal nonnegative.

d) We would like our model to only produce nonnegative temperatures. Consider a general linear dynamical system

$$\dot{x}(t) = Px(t)$$

For every node i we would like the quantity $x_i(t)$ to be nonnegative for all time. That is, if $x_i(0) \geq 0$ for all i, then

$$x_i(t) \geq 0 \qquad \text{for all } i \text{ and all } t \geq 0$$

Show that this implies that P is off-diagonal nonnegative.

e) Recall that a matrix P is called *Hurwitz* if all of its eigenvalues have negative real parts, and it is called *stable* if $e^{tP} \to 0$ as $t \to \infty$. A matrix is Hurwitz if and only if it is stable.

   Suppose P has the property that

$$e^{tP}x(0) \to 0 \quad \text{for all } x(0) \geq 0$$

   where $x(0) \geq 0$ means that all elements of $x(0)$ are nonnegative. Can we conclude that P is Hurwitz? If you think the answer is no, give an example, otherwise give a proof.

f) Suppose now that the source $s(t)$ is constant, so that the dynamics become

$$\dot{x}(t) = Qx(t) + s$$

   where $s \in \mathbb{R}^n$ is a vector which does not depend on time. Show that

$$Qx(t) = (e^{tQ} - I)s + Qe^{tQ}x(0)$$

3. Some true/false questions. For each statement, give a proof if it is true, or a counterexample if it is false.

a) Suppose $A \in \mathbb{R}^{n\times n}$ is invertible, and let $A = QR$ be its full QR factorization with Q orthogonal and R upper triangular. Let $B = RQ$. If A is symmetric, then B is symmetric.

b) Suppose $A \in \mathbb{R}^{n\times n}$ is invertible, and let $A = QR$ be its full QR factorization with Q orthogonal and R upper triangular. Let $B = RQ$. If A is upper triangular, then B is upper triangular.

c) Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Of all vectors x with $\|x\| = 1$, the x that maximizes $(x^T Ax)^2$ is the eigenvector corresponding to the largest eigenvalue λ_1.

d) Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Of all vectors x with $\|x\| = 1$, the x that minimizes $(x^T Ax)^2$ is the eigenvector corresponding to the smallest eigenvalue λ_n.

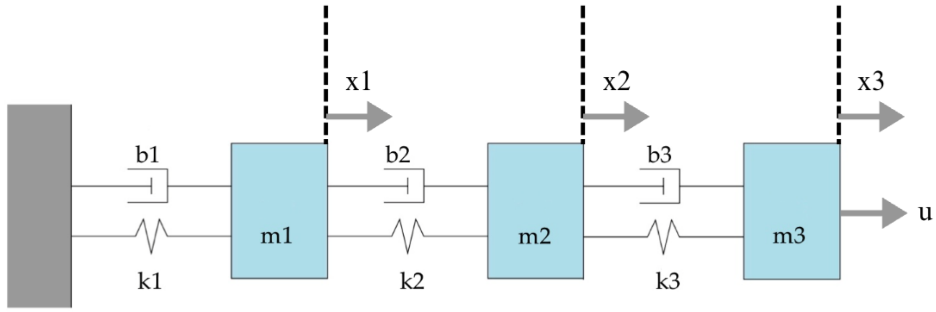
e) For any matrices A, B, C we have

$$\text{range}(ABC) \subseteq \text{range}(A)$$

f) $\|Ax\| \leq \|A\|_F \|x\|$ for all matrices $A \in \mathbb{R}^{m\times n}$ and vectors $x \in \mathbb{R}^n$. Here $\|A\|_F$ denotes the Frobenius norm.

g) If $A \in \mathbb{R}^{n\times n}$ has all eigenvalues λ satisfying $|\lambda| < 1$, then $\|A^2\| \leq \|A\|$.

4. Suspension system. Let us consider the following mechanical suspension system



where u is a force that applied to the third mass. The equations of motion for this system are

$$\begin{aligned}
\ddot{q}_1 &= -k_1 q_1 - b_1 \dot{q}_1 - k_2 q_1 + k_2 q_2 - b_2 \dot{q}_1 + b_2 \dot{q}_2 \\
\ddot{q}_2 &= -k_2 q_2 + k_2 q_1 - b_2 \dot{q}_2 + b_2 \dot{q}_1 - k_3 q_2 + k_3 q_3 - b_3 \dot{q}_2 + b_3 \dot{q}_3 \\
\ddot{q}_3 &= -k_3 q_3 + k_3 q_2 - b_3 \dot{q}_3 + b_3 \dot{q}_2 + u
\end{aligned}$$

a) We want to model the system as

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned}$$

Find A, B and C in terms of $k_1, k_2, k_3, b_1, b_2, b_3$. Use states $x = (q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$ and let $y = q_3$.

b) We will use sample period $h = 1$. Let $x_d(k) = x(kh)$, and $y_d(k) = y(kh)$. Suppose the force input u is piecewise constant, so that $u(t) = u_d(k)$ for $kh \leq t < kh + h$. Find a A_d, B_d and C_d so that these quantities satisfy the discrete-time linear dynamical system

$$\begin{aligned}
x_d(k+1) &= A_d x_d(k) + B_d u_d(k) \\
y_d(k) &= C_d x_d(k)
\end{aligned}$$

c) Suppose $x(0) = 0$. Use your discretization to find the response of the system when

$$u_d(k) = \begin{cases} 1 & \text{if } k < 3 \\ 0 & \text{otherwise} \end{cases}$$

Plot y against t for $0 \leq t \leq 100$.

d) Define the magnitude of the response J by

$$J = \sum_{k=0}^{99} y_d(k)^2$$

The force u_d is known to satisfy

$$\sum_{k=0}^{49} u_d(k)^2 \leq 1$$

and $u_d(k) = 0$ for $k \geq 50$. Find the the force u_d which satisfies these constraints and maximizes J. Report the maximum J, and plot the force input u_d, and the resulting output y_d.

5. Illumination. We have a square region R divided into $N \times N$ pixels. It is illuminated by n lamps. We apply power q_j to lamp j. In this question we will choose the vector of powers $q \in \mathbb{R}^n$ to illuminate the region R as uniformly as possible. The position of the ith lamp is given by the ith column of the matrix L, which is in the file `illumination.json`. We will use $N = 25$. The region R is

$$R = \{(x, y) \mid 0 \leq x \leq N, 0 \leq y \leq N\}$$

Suppose we apply power $a \in \mathbb{R}$ to a lamp at position $r \in \mathbb{R}^3$. Let $(x, y) \in \mathbb{R}^2$ be the center of one of the pixels in R. The distance from the lamp to the pixel is

$$d = \left((x - r_1)^2 + (y - r_2)^2 + r_3^2\right)^{\frac{1}{2}}$$

Then the light intensity falling on that pixel is ka/d^2. The total light intensity on any pixel is the sum of the intensities from the different lamps. We will have $k = 10$.

a) Let l_i be the light intensity on pixel i. We would like to apply power to the lamps to uniformly light the region R, so that each pixel receives light intensity approximately equal to 1. Define the error

$$J = \sum_{i=1}^{N^2} (l_i - 1)^2$$

Find the optimal choice of powers $q \in \mathbb{R}^n$, where q_j is the power applied to lamp j. Report the total power used

$$P_{\text{tot}} = \sum_{j=1}^{n} q_j$$

Plot the intensity as a *heatmap*. You may do this using

```
using Plots
heatmap(Y, c = :thermal, aspectratio = :equal, clims=(0,2))
```

where Y is an $N \times N$ matrix of intensities. Report the power applied to each lamp, and the error J.

b) We unfortunately have a power budget; we require $P_{\text{tot}} \le 15$. Here is one simple approach for achieving this. We use the optimal solution from the previous part, but scale it so that $P_{\text{tot}} = 15$. Do this, and report the error J.

c) It is possible to find the choice of powers that minimizes J subject to the budget constraint. Find this choice of powers. Plot the resulting heatmap of intensity, report the total power used, and the power applied to each lamp.

6. Finding missing data. We are given data from a ratings system, in the form of a matrix $Z \in \mathbb{R}^{m \times n}$. Here Z_{ij} is the rating that user i gives to movie j. A reasonable assumption about this setting is that such recommendation matrices have low rank. The idea is that a users rating depends on a few factors (that we don't know), such as whether they like romantic movies, comedies, etc. If there were r such hidden factors, then every row of the matrix would be a linear combination of r unknown vectors.

However, we do not have all entries of the matrix, because not every user has seen every movie. We would like to fill in the missing entries, making use of the low rank property of such matrices. We have

$$Z = \begin{bmatrix} X & P \\ R & Q \end{bmatrix}$$

where $X \in \mathbb{R}^{m_1 \times n_1}$ is the unknown block. In more realistic settings, the unknown entries would be distributed throughout the matrix. Here, for simplicity, we consider the case when all the top-left block is unknown. That means that the first m_1 users have not rated the first n_1 movies.

a) Show that

$$\operatorname{range} \begin{bmatrix} P \\ Q \end{bmatrix} \subseteq \operatorname{range} \begin{bmatrix} X & P \\ R & Q \end{bmatrix}$$

b) Now suppose that Q is fat and full rank. Show that there exists X such that

$$\operatorname{range} \begin{bmatrix} P \\ Q \end{bmatrix} = \operatorname{range} \begin{bmatrix} X & P \\ R & Q \end{bmatrix}$$

c) Using the results of the previous two parts, explain how to construct X which minimizes

$$\operatorname{rank} \begin{bmatrix} X & P \\ R & Q \end{bmatrix}$$

d) To save money, the movie streaming service decides to only show movies which have sufficiently high ratings. For each user i there is a threshhold rating s_i. The jth movie is shown if $Z_{ij} \geq s_i$ for all i.

Using the data in `movieratings.json`, find which movies are not shown.

Important note: you must solve this question exactly. Iterative solutions are not acceptable and will not receive points.