Matrix inverses in Julia

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Matrix inverses in Julia

- QR factorization
- inverse
- pseudo-inverse
- backslash operator
the qr command finds the QR factorization of a matrix
\[ A = \text{rand}(5, 3) \]
\[ Q, R = \text{qr}(A) \]

when columns of \( n \times k \) matrix \( A \) are independent, qr is same as ours

when columns are dependent, qr is not same as ours
- \( A = QR, Q^TQ = I \), and \( R_{ij} = 0 \) for \( i > j \) always holds
- \( R \) can have zero or negative diagonal entries
- \( R \) is not square when \( A \) is wide
Checking linear independence with Julia’s QR

- let’s check if columns of $A$ are linearly independent
- $A$ must be tall or square
- columns are linearly independent if and only if $R$ has no 0 diagonal entries
- check if columns of (tall or square) $A$ are linearly independent:
  
  ```julia
  a1 = rand(5)
  a2 = rand(5)
  A = [a1 a2 a1+a2] # linearly dependent columns
  Q, R = qr(A)
  # find the entry of diagonal of R closest to 0
  # R can have negative entries
  minimum(abs(diag(R)))
  ```
**Inverse**

- `inv(A)` returns the inverse matrix $A^{-1}$
- Julia will issue an error if
  - $A$ is not square
  - $A$ is not invertible
- You can solve square set of linear equations $Ax = b$, with invertible $A$, using
  
  ```
  b = rand(5,1)
  A = rand(5,5)
  x = inv(A)*b
  norm(A*x-b)  # check residual
  ```
  but there is a better way, using backslash
Pseudo-inverse

- for a $m \times n$ matrix $A$, $\text{pinv}(A)$ will return the $n \times m$ pseudo-inverse
- if $A$ is square and invertible
  - $\text{pinv}(A)$ will return the inverse $A^{-1}$
- if $A$ is tall with linearly independent columns
  - $\text{pinv}(A)$ will return the left inverse $(A^T A)^{-1} A^T$
- if $A$ is wide with linearly independent rows
  - $\text{pinv}(A)$ will return the right inverse $A^T (AA^T)^{-1}$
- in other cases, $\text{pinv}(A)$ returns an $m \times n$ matrix, but
  - it is not a left or right inverse of $A$
  - what it is is beyond the scope of this class
The backslash operator

- given $A$ and $b$, the \ operator solves the linear system $Ax = b$ for $x$
- for a $m \times n$ matrix $A$ and a $m$-vector $b$, $A\backslash b$ returns a $n$-vector $x$
- if $A$ is square and invertible
  - $x = A^{-1}b$
  - the unique solution of $Ax = b$
- if $A$ is tall with linearly independent columns
  - $x = (A^T A)^{-1} A^T b$
  - the least squares approximate solution of $Ax = b$
- if $A$ is wide with linearly independent rows
  - $x = A^T (AA^T)^{-1} b$
  - $x$ is the least norm solution of $Ax = b$
- in other cases, $A\backslash b$ returns an $n$-vector $x$, but what it means is beyond the scope of this class
- uses a factor and solve method similar to QR
Solving matrix systems with backslash

- solve matrix equation $AX = B$ for $X$, with $A$ square
- with $X = [x_1 \cdots x_k]$, $B = [b_1 \cdots b_k]$, same as solving $k$ linear systems

$$Ax_1 = b_1, \ldots, Ax_k = b_k$$

- $X = A\backslash B$ solves the system, doing the right thing:
  - factor $A$ once (order $n^3$)
  - back substitution to get $x_i = A^{-1}b_i, i = 1, \ldots, k$ (order $kn^2$)