Basic Notation

Basic set notation

\{a_1, \ldots, a_r\} \quad \text{the set with elements } a_1, \ldots, a_r.

\(a \in S\) \quad \text{a is in the set } S.

\(S = T\) \quad \text{the sets } S \text{ and } T \text{ are equal, i.e., every element of } S \text{ is in } T \text{ and every element of } T \text{ is in } S.

\(S \subseteq T\) \quad \text{the set } S \text{ is a subset of the set } T, \text{ i.e., every element of } S \text{ is also an element of } T.

\exists a \in S \ \mathcal{P}(a) \quad \text{there exists an } a \text{ in } S \text{ for which the property } \mathcal{P} \text{ holds.}

\forall x \in S \ \mathcal{P}(a) \quad \text{property } \mathcal{P} \text{ holds for every element in } S.

\{a \in S \mid \mathcal{P}(a)\} \quad \text{the set of all } a \text{ in } S \text{ for which } \mathcal{P} \text{ holds (the set } S \text{ is sometimes omitted if it can be determined from context).}

\(A \cup B\) \quad \text{union of sets, } A \cup B = \{x \mid x \in A \text{ or } x \in B\}.

\(A \cap B\) \quad \text{intersection of sets, } A \cap B = \{x \mid x \in A \text{ and } x \in B\}.

\(A \times B\) \quad \text{Cartesian product of two sets, } A \times B = \{(a, b) \mid a \in A, \ b \in B\}.

Some specific sets

\(\mathbb{R}\) \quad \text{the set of real numbers.}

\(\mathbb{R}^n\) \quad \text{the set of real } n\text{-vectors (}\ n \times 1 \text{ matrices).}

\(\mathbb{R}^{1 \times n}\) \quad \text{the set of real } n\text{-row-vectors (}\ 1 \times n \text{ matrices).}

\(\mathbb{R}^{m \times n}\) \quad \text{the set of real } m \times n \text{ matrices.}

\(j\) \quad \text{can mean } \sqrt{-1}, \text{ in the company of electrical engineers.}

\(i\) \quad \text{can mean } \sqrt{-1}, \text{ for normal people; } i \text{ is the polite term in mixed company (i.e., when non-electrical engineers are present).}

\(\mathbb{C}, \mathbb{C}^n, \mathbb{C}^{m \times n}\) \quad \text{the set of complex numbers, complex } n\text{-vectors, complex } m \times n \text{ matrices.}

\(\mathbb{Z}\) \quad \text{the set of integers: } \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}.

\(\mathbb{R}_+\) \quad \text{the nonnegative real numbers, i.e., } \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}.

\([a, b], (a, b), [a, b), (a, b)\) \quad \text{the real intervals } \{x \mid a \leq x \leq b\}, \{x \mid a < x \leq b\}, \{x \mid a \leq x < b\}, \text{ and } \{x \mid a < x < b\}, \text{ respectively.}
Vectors and matrices

We use square brackets \([\) and \(]\) to construct matrices and vectors, with white space delineating the entries in a row, and a new line indicating a new row. For example \([1 \ 2]\) is a row vector in \(\mathbb{R}^{1\times2}\), and \([1 \ 2 \ 3 \ 4 \ 5 \ 6]\) is matrix in \(\mathbb{R}^{2\times3}\). \([1 \ 2]\)\(^T\) denotes a column vector, \(i.e.,\) an element of \(\mathbb{R}^{2\times1}\), which we abbreviate as \(\mathbb{R}^2\).

We use curved brackets \((\) and \(])\) surrounding lists of entries, delineated by commas, as an alternative method to construct (column) vectors. Thus, we have three ways to denote a column vector:

\[
(1, 2) = [1 \ 2] = \begin{bmatrix}
1 \\
2
\end{bmatrix}.
\]

Note that in our notation scheme (which is fairly standard), \([1, 2, 3]\) and \((1 \ 2 \ 3)\) aren’t used. We also use square and curved brackets to construct block matrices and vectors. For example if \(x, y, z \in \mathbb{R}^n\), we have

\[
\begin{bmatrix}
x & y & z
\end{bmatrix} \in \mathbb{R}^{n\times3},
\]

a matrix with columns \(x, y,\) and \(z\). We can construct a block vector as

\[
(x, y, z) = \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} \in \mathbb{R}^{3n}.
\]

Functions

The notation \(f: A \to B\) means that \(f\) is a function on the set \(A\) into the set \(B\). The notation \(b = f(a)\) means \(b\) is the value of the function \(f\) at the point \(a\), where \(a \in A\) and \(b \in B\). The set \(A\) is called the domain of the function \(f\); it can be thought of as the set of legal parameter values that can be passed to the function \(f\). The set \(B\) is called the codomain (or sometimes range) of the function \(f\); it can be thought of as the set that contains all possible returned values of the function \(f\).

There are several ways to think of a function. The formal definition is that \(f\) is a subset of \(A \times B\), with the property that for every \(a \in A\), there is exactly one \(b \in B\) such that \((a, b) \in f\). We denote this as \(b = f(a)\).

Perhaps a better way to think of a function is as a black box or (software) function or subroutine. The domain is the set of all legal values (or data types or structures) that can be passed to \(f\). The codomain of \(f\) gives the data type or data structure of the values returned by \(f\).

Thus \(f(a)\) is meaningless if \(a \notin A\). If \(a \in A\), then \(b = f(a)\) is an element of \(B\). Also note that the function is denoted \(f\); it is wrong to say ‘the function \(f(a)\)’ (since \(f(a)\) is an element
of \( B \), not a function). Having said that, we do sometimes use sloppy notation such as ‘the function \( f(t) = t^3 \)’. To say this more clearly you could say ‘the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(t) = t^3 \) for \( t \in \mathbb{R} \).

Examples

- \(-0.1 \in \mathbb{R}, \sqrt{2} \in \mathbb{R}_+, 1 - 2j \in \mathbb{C} \) (with \( j = \sqrt{-1} \)).
- The matrix
  \[
  A = \begin{bmatrix}
  0.3 & 6.1 & -0.12 \\
  7.2 & 0 & 0.01
  \end{bmatrix}
  \]
is an element in \( \mathbb{R}^{2 \times 3} \). We can define a function \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) as \( f(x) = Ax \) for any \( x \in \mathbb{R}^3 \). If \( x \in \mathbb{R}^3 \), then \( f(x) \) is a particular vector in \( \mathbb{R}^2 \). We can say ‘the function \( f \) is linear’. To say ‘the function \( f(x) \) is linear’ is technically wrong since \( f(x) \) is a vector, not a function. Similarly we can’t say ‘\( A \) is linear’; it is just a matrix.
- We can define a function \( f : \{a \in \mathbb{R} \mid a \neq 0\} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) by \( f(a, x) = (1/a)x \), for any nonzero \( a \in \mathbb{R} \), and any \( x \in \mathbb{R}^n \). The function \( f \) could be informally described as division of a vector by a nonzero scalar.
- Consider the set \( A = \{0, -1, 3.2\} \). The elements of \( A \) are 0, -1 and 3.2. Therefore, for example, \(-1 \in A \) and \( \{0, 3.2\} \subseteq A \). Also, we can say that \( \forall x \in A, -1 \leq x \leq 4 \) or \( \exists x \in A, x > 3 \).
- Suppose \( A = \{1, -1\} \). Another representation for \( A \) is \( A = \{x \in \mathbb{R} \mid x^2 = 1\} \).
- Suppose \( A = \{1, -2, 0\} \) and \( B = \{3, -2\} \). Then
  \[
  A \cup B = \{1, -2, 0, 3\}, \quad A \cap B = \{-2\}.
  \]
- Suppose \( A = \{1, -2, 0\} \) and \( B = \{1, 3\} \). Then
  \[
  A \times B = \{(1,1), (1,3), (-2,1), (-2,3), (0,1), (0,3)\}.
  \]
- \( f : \mathbb{R} \rightarrow \mathbb{R} \) with \( f(x) = x^2 - x \) defines a function from \( \mathbb{R} \) to \( \mathbb{R} \) while \( u : \mathbb{R}_+ \rightarrow \mathbb{R}^2 \) with
  \[
  u(t) = \begin{bmatrix}
  t \cos t \\
  2e^{-t}
  \end{bmatrix}
  \]
defines a function from \( \mathbb{R}_+ \) to \( \mathbb{R}^2 \).