Symmetric matrices and quadratic forms

- eigenvectors of symmetric matrices
- quadratic forms
- inequalities for quadratic forms
- positive semidefinite matrices

Eigenvalues of symmetric matrices

if $A \in \mathbb{R}^{n \times n}$ is symmetric, *i.e.*, $A = A^\mathsf{T}$, then the eigenvalues of A are real

to see this, suppose $Av = \lambda v$, $v \neq 0$, $v \in \mathbb{C}^n$, then

$$\overline{v}^\mathsf{T} A v = \overline{v}^\mathsf{T} (Av) = \lambda \overline{v}^\mathsf{T} v = \lambda \sum_{i=1}^{n} |v_i|^2$$

but also

$$\overline{v}^\mathsf{T} A v = \overline{(Av)}^\mathsf{T} v = \overline{(\lambda v)}^\mathsf{T} v = \overline{\lambda} \sum_{i=1}^{n} |v_i|^2$$

so we have $\lambda = \overline{\lambda}$, *i.e.*, $\lambda \in \mathbb{R}$ (hence, can assume $v \in \mathbb{R}^n$)

Eigenvectors of symmetric matrices

there is a set of n orthonormal eigenvectors of A

- *i.e.*, $q_1, \ldots, q_n$ s.t. $Aq_i = \lambda_i q_i$, $q_i^\mathsf{T} q_j = \delta_{ij}$
- in matrix form: there is an orthogonal Q s.t.

$$Q^{-1}AQ = Q^\mathsf{T}AQ = \Lambda$$

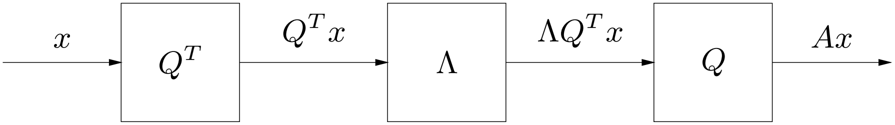
- hence we can express A as

$$A = Q\Lambda Q^\mathsf{T} = \sum_{i=1}^{n} \lambda_i q_i q_i^\mathsf{T}$$

- in particular, q_i are both left and right eigenvectors

Interpretations

$A = Q\Lambda Q^\mathsf{T}$ corresponds to



linear mapping $y = Ax$ can be decomposed as

- resolve into q_i coordinates
- scale coordinates by λ_i
- reconstitute with basis q_i

Geometrical interpretation

multiplication by A is the same as

- rotate by Q^{T}
- diagonal real scale ('dilation') by Λ
- rotate back by Q

decomposition

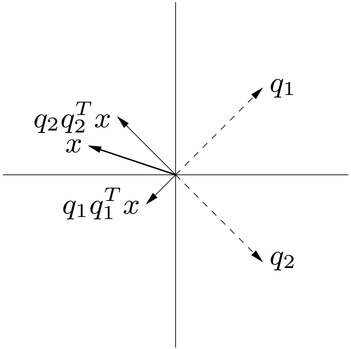
$$A = \sum_{i=1}^{n} \lambda_i q_i q_i^{\mathsf{T}}$$

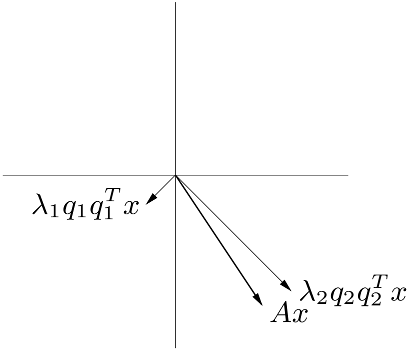
expresses A as linear combination of 1-dimensional projections

Example:

$$A = \begin{bmatrix} -1/2 & 3/2 \\ 3/2 & -1/2 \end{bmatrix}$$

$$= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^{\mathsf{T}}$$





Proof

eigenvectors corresponding to distinct eigenvalues are orthogonal

- since λ_i distinct, have $v_1, \ldots, v_n$, a set of linearly independent eigenvectors of A

$$Av_i = \lambda_i v_i, \qquad ||v_i|| = 1$$

- then $v_i^\mathsf{T}(Av_j) = \lambda_j v_i^\mathsf{T} v_j = (Av_i)^\mathsf{T} v_j = \lambda_i v_i^\mathsf{T} v_j$
- and $(\lambda_i - \lambda_j) v_i^\mathsf{T} v_j = 0$
- for $i \neq j$, $\lambda_i \neq \lambda_j$, hence $v_i^\mathsf{T} v_j = 0$
- in this case we can say: eigenvectors *are* orthogonal
- in general case (λ_i not distinct) we must say: eigenvectors *can be chosen* to be orthogonal

Quadratic forms

a *quadratic form* is a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = x^T A x = \sum_{i,j=1}^{n} A_{ij} x_i x_j$$

- in a quadratic form we may as well assume $A = A^T$ since

  $$x^T A x = x^T ((A + A^T)/2) x$$

  ($(A + A^T)/2$ is called the *symmetric part* of A)

- **uniqueness:** if $x^T A x = x^T B x$ for all $x \in \mathbb{R}^n$ and $A = A^T$, $B = B^T$, then $A = B$

Examples

quadratic forms

- $||Bx||^2 = x^\mathsf{T} B^\mathsf{T} B x$
- $\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2$
- $||Fx||^2 - ||Gx||^2$

sets defined by quadratic forms:

- $\{ \, x \mid f(x) = a \, \}$ is called a *quadratic surface*
- $\{ \, x \mid f(x) \leq a \, \}$ is called a *quadratic region*

Inequalities for quadratic forms

suppose $A = A^T$, $A = Q\Lambda Q^T$ with eigenvalues sorted so $\lambda_1 \geq \cdots \geq \lambda_n$ then

$$\boxed{x^T A x \leq \lambda_1 x^T x}$$

because

$$\begin{aligned}
x^T A x &= x^T Q \Lambda Q^T x \\
&= (Q^T x)^T \Lambda (Q^T x) \\
&= \sum_{i=1}^{n} \lambda_i (q_i^T x)^2 \\
&\leq \lambda_1 \sum_{i=1}^{n} (q_i^T x)^2 \\
&= \lambda_1 ||x||^2
\end{aligned}$$

Inequalities

- similar argument shows $x^\mathsf{T} Ax \geq \lambda_n \|x\|^2$, so we have

$$\lambda_n x^\mathsf{T} x \leq x^\mathsf{T} Ax \leq \lambda_1 x^\mathsf{T} x$$

- sometimes λ_1 is called $\lambda_{\max}$, λ_n is called $\lambda_{\min}$

- note also that

$$q_1^\mathsf{T} Aq_1 = \lambda_1 \|q_1\|^2, \qquad q_n^\mathsf{T} Aq_n = \lambda_n \|q_n\|^2,$$

  so the inequalities are tight

Positive semidefinite and positive definite matrices

suppose $A = A^{\mathsf{T}} \in \mathbb{R}^{n \times n}$

we say A is **positive semidefinite** if $x^{\mathsf{T}} A x \geq 0$ for all x

- *this is written $A \geq 0$* (and sometimes $A \succeq 0$)
- $A \geq 0$ if and only if $\lambda_{\min}(A) \geq 0$, $i.e.$, all eigenvalues are nonnegative
- **not** the same as $A_{ij} \geq 0$ for all i, j

we say A is *positive definite* if $x^{\mathsf{T}} A x > 0$ for all $x \neq 0$

- denoted $A > 0$
- $A > 0$ if and only if $\lambda_{\min}(A) > 0$, $i.e.$, all eigenvalues are positive

Matrix inequalities

- we say A is *negative semidefinite* if $-A \geq 0$
- we say A is *negative definite* if $-A > 0$
- otherwise, we say A is *indefinite*

matrix inequality: if A and B are both symmetric, we use $A < B$ to mean $B - A > 0$.

- many variations, for example $A \geq B$ means $A - B \geq 0$,
- $A > B$ means $x^\mathsf{T} A x > x^\mathsf{T} B x$ for all $x \neq 0$

Matrix inequalities

many properties that you'd guess hold actually do, $e.g.$,

- if $A \geq B$ and $C \geq D$, then $A + C \geq B + D$
- if $B \leq 0$ then $A + B \leq A$
- if $A \geq 0$ and $\alpha \geq 0$, then $\alpha A \geq 0$
- $A^2 \geq 0$
- if $A > 0$, then $A^{-1} > 0$

matrix inequality is only a *partial order*: we can have

$$A \not\geq B, \qquad B \not\geq A$$

(such matrices are called *incomparable*)