Symmetric matrices and quadratic forms

- eigenvectors of symmetric matrices
- quadratic forms
- inequalities for quadratic forms
- positive semidefinite matrices

Eigenvalues of symmetric matrices

if $A \in \mathbb{R}^{n\times n}$ is symmetric, *i.e.*, $A = A^\mathsf{T}$, then the eigenvalues of A are real

to see this, suppose $Av = \lambda v$, $v \neq 0$, $v \in \mathbb{C}^n$, then

$$\overline{v}^\mathsf{T} A v = \overline{v}^\mathsf{T}(Av) = \lambda \overline{v}^\mathsf{T} v = \lambda \sum_{i=1}^n |v_i|^2$$

but also

$$\overline{v}^\mathsf{T} A v = \overline{(Av)}^\mathsf{T} v = \overline{(\lambda v)}^\mathsf{T} v = \overline{\lambda} \sum_{i=1}^n |v_i|^2$$

so we have $\lambda = \overline{\lambda}$, *i.e.*, $\lambda \in \mathbb{R}$ (hence, can assume $v \in \mathbb{R}^n$)

Eigenvectors of symmetric matrices

there is a set of n orthonormal eigenvectors of A

- *i.e.*, $q_1, \ldots, q_n$ s.t. $Aq_i = \lambda_i q_i$, $q_i^\mathsf{T} q_j = \delta_{ij}$
- in matrix form: there is an orthogonal Q s.t.

$$Q^{-1}AQ = Q^\mathsf{T}AQ = \Lambda$$

- hence we can express A as

$$A = Q\Lambda Q^\mathsf{T} = \sum_{i=1}^{n} \lambda_i q_i q_i^\mathsf{T}$$

- in particular, q_i are both left and right eigenvectors

Interpretations

$A = Q\Lambda Q^{\mathsf{T}}$ corresponds to

$$\xrightarrow{\;x\;} \boxed{Q^T} \xrightarrow{\;Q^T x\;} \boxed{\Lambda} \xrightarrow{\;\Lambda Q^T x\;} \boxed{Q} \xrightarrow{\;Ax\;}$$

linear mapping $y = Ax$ can be decomposed as

- resolve into q_i coordinates
- scale coordinates by λ_i
- reconstitute with basis q_i

Geometrical interpretation

multiplication by A is the same as

- rotate by Q^T
- diagonal real scale ('dilation') by Λ
- rotate back by Q

decomposition

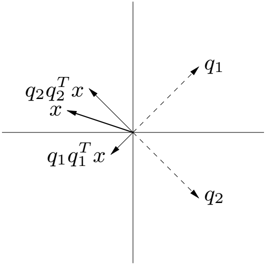
$$A = \sum_{i=1}^{n} \lambda_i q_i q_i^\mathsf{T}$$

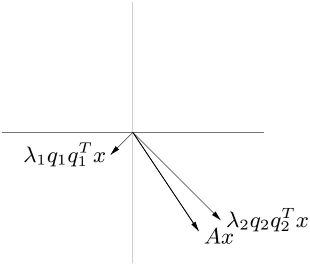
expresses A as linear combination of 1-dimensional projections

Example:

$$A = \begin{bmatrix} -1/2 & 3/2 \\ 3/2 & -1/2 \end{bmatrix}$$

$$= \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)^{\mathsf{T}}$$





Proof

eigenvectors corresponding to distinct eigenvalues are orthogonal

- since λ_i distinct, have $v_1, \ldots, v_n$, a set of linearly independent eigenvectors of A

$$Av_i = \lambda_i v_i, \qquad \|v_i\| = 1$$

- then $v_i^\mathsf{T}(Av_j) = \lambda_j v_i^\mathsf{T} v_j = (Av_i)^\mathsf{T} v_j = \lambda_i v_i^\mathsf{T} v_j$
- and $(\lambda_i - \lambda_j) v_i^\mathsf{T} v_j = 0$
- for $i \neq j$, $\lambda_i \neq \lambda_j$, hence $v_i^\mathsf{T} v_j = 0$
- in this case we can say: eigenvectors *are* orthogonal
- in general case (λ_i not distinct) we must say: eigenvectors *can be chosen* to be orthogonal

Quadratic forms

a *quadratic form* is a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = x^\mathsf{T} A x = \sum_{i,j=1}^{n} A_{ij} x_i x_j$$

- in a quadratic form we may as well assume $A = A^\mathsf{T}$ since

  $$x^\mathsf{T} A x = x^\mathsf{T}((A + A^\mathsf{T})/2)x$$

  ($(A + A^\mathsf{T})/2$ is called the *symmetric part* of A)
- **uniqueness:** if $x^\mathsf{T} A x = x^\mathsf{T} B x$ for all $x \in \mathbb{R}^n$ and $A = A^\mathsf{T}$, $B = B^\mathsf{T}$, then $A = B$

Examples

quadratic forms

- $\|Bx\|^2 = x^\mathsf{T} B^\mathsf{T} B x$
- $\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2$
- $\|Fx\|^2 - \|Gx\|^2$

sets defined by quadratic forms:

- $\{ x \mid f(x) = a \}$ is called a *quadratic surface*
- $\{ x \mid f(x) \le a \}$ is called a *quadratic region*

Inequalities for quadratic forms

suppose $A = A^\mathsf{T}$, $A = Q\Lambda Q^\mathsf{T}$ with eigenvalues sorted so $\lambda_1 \geq \cdots \geq \lambda_n$ then

$$\boxed{x^\mathsf{T} A x \leq \lambda_1 x^\mathsf{T} x}$$

because

$$\begin{aligned} x^\mathsf{T} A x &= x^\mathsf{T} Q\Lambda Q^\mathsf{T} x \\ &= (Q^\mathsf{T} x)^\mathsf{T} \Lambda (Q^\mathsf{T} x) \\ &= \sum_{i=1}^n \lambda_i (q_i^\mathsf{T} x)^2 \\ &\leq \lambda_1 \sum_{i=1}^n (q_i^\mathsf{T} x)^2 \\ &= \lambda_1 \|x\|^2 \end{aligned}$$

Inequalities

- similar argument shows $x^{\mathsf{T}}Ax \geq \lambda_n \|x\|^2$, so we have

$$\lambda_n x^{\mathsf{T}}x \leq x^{\mathsf{T}}Ax \leq \lambda_1 x^{\mathsf{T}}x$$

- sometimes λ_1 is called $\lambda_{\max}$, λ_n is called $\lambda_{\min}$
- note also that

$$q_1^{\mathsf{T}}Aq_1 = \lambda_1 \|q_1\|^2, \qquad q_n^{\mathsf{T}}Aq_n = \lambda_n \|q_n\|^2,$$

so the inequalities are tight

Positive semidefinite and positive definite matrices

suppose $A = A^\mathsf{T} \in \mathbb{R}^{n\times n}$

we say A is **positive semidefinite** if $x^\mathsf{T} Ax \geq 0$ for all x

- *this is written* $A \geq 0$ (and sometimes $A \succeq 0$)
- $A \geq 0$ if and only if $\lambda_{\mathrm{min}}(A) \geq 0$, *i.e.*, all eigenvalues are nonnegative
- **not** the same as $A_{ij} \geq 0$ for all i, j

we say A is *positive definite* if $x^\mathsf{T} Ax > 0$ for all $x \neq 0$

- denoted $A > 0$
- $A > 0$ if and only if $\lambda_{\mathrm{min}}(A) > 0$, *i.e.*, all eigenvalues are positive

Matrix inequalities

- we say A is *negative semidefinite* if $-A \geq 0$
- we say A is *negative definite* if $-A > 0$
- otherwise, we say A is *indefinite*

matrix inequality: if A and B are both symmetric, we use $A < B$ to mean $B - A > 0$.

- many variations, for example $A \geq B$ means $A - B \geq 0$,
- $A > B$ means $x^\mathsf{T} A x > x^\mathsf{T} B x$ for all $x \neq 0$

Matrix inequalities

many properties that you'd guess hold actually do, $e.g.$,

- if $A \geq B$ and $C \geq D$, then $A + C \geq B + D$
- if $B \leq 0$ then $A + B \leq A$
- if $A \geq 0$ and $\alpha \geq 0$, then $\alpha A \geq 0$
- $A^2 \geq 0$
- if $A > 0$, then $A^{-1} > 0$

matrix inequality is only a *partial order*: we can have

$$A \not\geq B, \qquad B \not\geq A$$

(such matrices are called *incomparable*)