Recursive estimation

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Growing sets of measurements

least-squares problem in 'row' form

$$\text{minimize} \quad \|Ax - y\|^2 = \sum_{i=1}^{m} (\tilde{a}_i^\mathsf{T} x - y_i)^2$$

where $\tilde{a}_i^\mathsf{T}$ are the *rows* of A ($\tilde{a}_i \in \mathbb{R}^n$)

- $x \in \mathbb{R}^n$ is some vector to be estimated
- each pair $\tilde{a}_i$, y_i corresponds to one measurement
- solution is
$$x_{\mathsf{ls}} = \left(\sum_{i=1}^{m} \tilde{a}_i \tilde{a}_i^\mathsf{T} \right)^{-1} \sum_{i=1}^{m} y_i \tilde{a}_i$$
- *recursive estimation:* $\tilde{a}_i$ and y_i become available sequentially, $i.e.$, m increases with time

Recursive least-squares

we can compute $x_{\mathrm{ls}}(m) = \left(\sum_{i=1}^{m} \tilde{a}_i \tilde{a}_i^T\right)^{-1} \sum_{i=1}^{m} y_i \tilde{a}_i$ recursively

the algorithm is

$$
\begin{aligned}
&P(0) = 0 \in \mathbb{R}^{n\times n} \\
&q(0) = 0 \in \mathbb{R}^{n} \\
&\textbf{for } m = 0, 1, \ldots, \\
&\qquad P(m+1) = P(m) + \tilde{a}_{m+1}\tilde{a}_{m+1}^T \\
&\qquad q(m+1) = q(m) + y_{m+1} a_{m+1}
\end{aligned}
$$

- if $P(m)$ is invertible, we have $x_{\mathrm{ls}}(m) = P(m)^{-1} q(m)$
- $P(m)$ is invertible $\iff$ $\tilde{a}_1, \ldots, \tilde{a}_m$ span $\mathbb{R}^n$
  (so, once $P(m)$ becomes invertible, it stays invertible)

Fast update for recursive least-squares

we can calculate

$$P(m+1)^{-1} = \left(P(m) + \tilde{a}_{m+1}\tilde{a}_{m+1}^{\mathsf{T}}\right)^{-1}$$

efficiently from $P(m)^{-1}$ using the *rank one update formula*

$$\left(P + aa^{\mathsf{T}}\right)^{-1} = P^{-1} - \frac{1}{1 + a^{\mathsf{T}}P^{-1}a}(P^{-1}a)(P^{-1}a)^{\mathsf{T}}$$

- valid when $P = P^{\mathsf{T}}$, and P and $P + aa^{\mathsf{T}}$ are both invertible
- gives an $O(n^2)$ method for computing $P(m+1)^{-1}$ from $P(m)^{-1}$
- standard methods for computing $P(m+1)^{-1}$ from $P(m+1)$ are $O(n^3)$

Verification of rank one update formula

$$
\begin{aligned}
(P + aa^\mathsf{T})\left(P^{-1} - \frac{1}{1 + a^\mathsf{T}P^{-1}a}(P^{-1}a)(P^{-1}a)^\mathsf{T}\right) \\
&= I + aa^\mathsf{T}P^{-1} - \frac{1}{1 + a^\mathsf{T}P^{-1}a}P(P^{-1}a)(P^{-1}a)^\mathsf{T} \\
&\qquad - \frac{1}{1 + a^\mathsf{T}P^{-1}a}aa^\mathsf{T}(P^{-1}a)(P^{-1}a)^\mathsf{T} \\
&= I + aa^\mathsf{T}P^{-1} - \frac{1}{1 + a^\mathsf{T}P^{-1}a}aa^\mathsf{T}P^{-1} - \frac{a^\mathsf{T}P^{-1}a}{1 + a^\mathsf{T}P^{-1}a}aa^\mathsf{T}P^{-1} \\
&= I
\end{aligned}
$$