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Recursive estimation

Growing sets of measurements

least-squares problem in 'row' form

$$\text{minimize} \quad \|Ax - y\|^2 = \sum_{i=1}^{m} (a_i^T x - y_i)^2$$

where a_i^T are the *rows* of A ($a_i \in \mathbb{R}^n$)

- $x \in \mathbb{R}^n$ is some vector to be estimated
- each pair a_i, y_i corresponds to one measurement
- solution is
$$x_{\text{ls}} = \left(\sum_{i=1}^{m} a_i a_i^T \right)^{-1} \sum_{i=1}^{m} y_i a_i$$
- *recursive estimation:* a_i and y_i become available sequentially, $i.e.$, m increases with time

Recursive least-squares

we can compute $x_{\text{ls}}(m) = \left(\sum_{i=1}^{m} a_i a_i^T \right)^{-1} \sum_{i=1}^{m} y_i a_i$ recursively

the algorithm is

$$
\begin{aligned}
& P(0) = 0 \in \mathbb{R}^{n \times n} \\
& q(0) = 0 \in \mathbb{R}^{n} \\
& \textbf{for } m = 0, 1, \ldots, \\
& \qquad P(m+1) = P(m) + a_{m+1} a_{m+1}^T \\
& \qquad q(m+1) = q(m) + y_{m+1} a_{m+1}
\end{aligned}
$$

- if $P(m)$ is invertible, we have $x_{\text{ls}}(m) = P(m)^{-1} q(m)$
- $P(m)$ is invertible $\iff$ $a_1, \ldots, a_m$ span $\mathbb{R}^n$
  (so, once $P(m)$ becomes invertible, it stays invertible)

Fast update for recursive least-squares

we can calculate

$$P(m+1)^{-1} = \left(P(m) + a_{m+1}a_{m+1}^{\mathsf{T}}\right)^{-1}$$

efficiently from $P(m)^{-1}$ using the *rank one update formula*

$$\boxed{\left(P + aa^{\mathsf{T}}\right)^{-1} = P^{-1} - \frac{1}{1 + a^{\mathsf{T}}P^{-1}a}(P^{-1}a)(P^{-1}a)^{\mathsf{T}}}$$

- valid when $P = P^{\mathsf{T}}$, and P and $P + aa^{\mathsf{T}}$ are both invertible
- gives an $O(n^2)$ method for computing $P(m+1)^{-1}$ from $P(m)^{-1}$
- standard methods for computing $P(m+1)^{-1}$ from $P(m+1)$ are $O(n^3)$

Verification of rank one update formula

$$
\begin{aligned}
(P + aa^\mathsf{T}) & \left(P^{-1} - \frac{1}{1 + a^\mathsf{T} P^{-1} a} (P^{-1} a)(P^{-1} a)^\mathsf{T} \right) \\
&= I + aa^\mathsf{T} P^{-1} - \frac{1}{1 + a^\mathsf{T} P^{-1} a} P (P^{-1} a)(P^{-1} a)^\mathsf{T} \\
&\qquad - \frac{1}{1 + a^\mathsf{T} P^{-1} a} aa^\mathsf{T} (P^{-1} a)(P^{-1} a)^\mathsf{T} \\
&= I + aa^\mathsf{T} P^{-1} - \frac{1}{1 + a^\mathsf{T} P^{-1} a} aa^\mathsf{T} P^{-1} - \frac{a^\mathsf{T} P^{-1} a}{1 + a^\mathsf{T} P^{-1} a} aa^\mathsf{T} P^{-1} \\
&= I
\end{aligned}
$$