Rank

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Rank of a matrix

we define the *rank* of $A \in \mathbb{R}^{m \times n}$ as

$$\boxed{\mathbf{rank}(A) = \mathbf{dim}\,\mathbf{range}(A)}$$

(nontrivial) **facts:**

- $\mathbf{rank}(A) = \mathbf{rank}(A^T)$
- $\mathbf{rank}(A)$ is maximum number of independent columns (or rows) of A
  hence $\mathbf{rank}(A) \leq \mathbf{min}(m, n)$

Rank of a matrix

- **rank**(A) is maximum number of independent columns of A
  - to see this, notice that if the columns of A are independent, then the number of columns r is the rank, since the columns are a basis for the range
  - and if not, then there must be one column in the span of the others, so remove it, and repeat if necessary
  - all other independent sets of columns must have no more than r elements.
- proof of **rank**$(A) =$ **rank**(A^T) uses QR, to come

Conservation of dimension

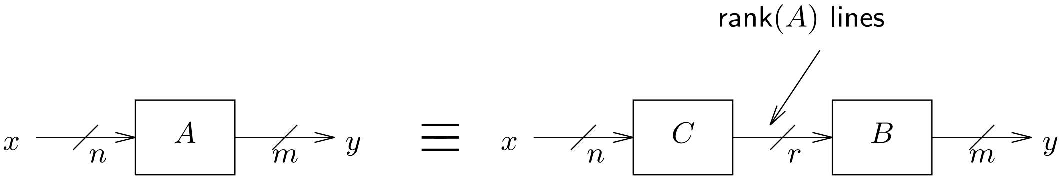
$$\boxed{\textbf{dim range}(A) + \textbf{dim null}(A) = n}$$

- **rank**(A) is dimension of set 'hit' by the mapping $y = Ax$
- **dim null**(A) is dimension of set of x 'crushed' to zero by $y = Ax$
- 'conservation of dimension': each dimension of input is either crushed to zero or ends up in output
- roughly speaking:
  - n is number of degrees of freedom in input x
  - **dim null**(A) is number of degrees of freedom lost in the mapping from x to $y = Ax$
  - **rank**(A) is number of degrees of freedom in output y
- proof using QR

Coding interpretation of rank

$$\textbf{rank}(BC) \le \textbf{min}\{\textbf{rank}(B), \textbf{rank}(C)\}$$

- hence if $A = BC$ with $B \in \mathbb{R}^{m\times r}$, $C \in \mathbb{R}^{r\times n}$, then $\textbf{rank}(A) \le r$
- converse: if $\textbf{rank}(A) = r$ then $A \in \mathbb{R}^{m\times n}$ factors as $A = BC$ with $B \in \mathbb{R}^{m\times r}$, $C \in \mathbb{R}^{r\times n}$
- $\textbf{rank}(A) = r$ is minimum size vector needed to faithfully reconstruct y from x



Coding interpretation of rank

- **rank**$(BC) \le$ **rank**(B) because **range**$(BC) \subset$ **range**(B)
- transpose implies **rank**$(BC) \le$ **rank**(C)
- factorization converse comes from QR

Application: fast matrix-vector multiplication

- need to compute matrix-vector product $y = Ax$, $A \in \mathbb{R}^{m \times n}$
- A has known factorization $A = BC$, $B \in \mathbb{R}^{m \times r}$
- computing $y = Ax$ directly: mn operations
- computing $y = Ax$ as $y = B(Cx)$ (compute $z = Cx$ first, then $y = Bz$): $rn + mr = (m + n)r$ operations
- savings can be considerable if $r \ll \mathbf{min}\{m, n\}$

Full rank matrices

for $A \in \mathbb{R}^{m \times n}$ we always have $\mathbf{rank}(A) \leq \mathbf{min}(m, n)$

we say A is *full rank* if $\mathbf{rank}(A) = \mathbf{min}(m, n)$

- for **square** matrices, full rank means nonsingular
- for **skinny** matrices ($m \geq n$), full rank means columns are independent
- for **fat** matrices ($m \leq n$), full rank means rows are independent