Rank

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Rank of a matrix

we define the *rank* of $A \in \mathbb{R}^{m \times n}$ as

$$\text{rank}(A) = \dim \text{range}(A)$$

(nontrivial) facts:

- $\text{rank}(A) = \text{rank}(A^T)$
- $\text{rank}(A)$ is maximum number of independent columns (or rows) of $A$ hence $\text{rank}(A) \leq \min(m, n)$
Rank of a matrix

- \( \text{rank}(A) \) is maximum number of independent columns of \( A \)
  
  - to see this, notice that if the columns of \( A \) are independent, then the number of columns \( r \) is the rank, since the columns are a basis for the range
  
  - and if not, then there must be one column in the span of the others, so remove it, and repeat if necessary
  
  - all other independent sets of columns must have no more than \( r \) elements.

- proof of \( \text{rank}(A) = \text{rank}(A^T) \) uses QR, to come
Conservation of dimension

\[ \text{dim range}(A) + \text{dim null}(A) = n \]

- \( \text{rank}(A) \) is dimension of set ‘hit’ by the mapping \( y = Ax \)
- \( \text{dim null}(A) \) is dimension of set of \( x \) ‘crushed’ to zero by \( y = Ax \)
- ‘conservation of dimension’: each dimension of input is either crushed to zero or ends up in output
- roughly speaking:
  - \( n \) is number of degrees of freedom in input \( x \)
  - \( \text{dim null}(A) \) is number of degrees of freedom lost in the mapping from \( x \) to \( y = Ax \)
  - \( \text{rank}(A) \) is number of degrees of freedom in output \( y \)
- proof using QR
Coding interpretation of rank

\[ \text{rank}(BC) \leq \min\{\text{rank}(B), \text{rank}(C)\} \]

- hence if \( A = BC \) with \( B \in \mathbb{R}^{m \times r} \), \( C \in \mathbb{R}^{r \times n} \), then \( \text{rank}(A) \leq r \)

- converse: if \( \text{rank}(A) = r \) then \( A \in \mathbb{R}^{m \times n} \) factors as \( A = BC \) with \( B \in \mathbb{R}^{m \times r} \), \( C \in \mathbb{R}^{r \times n} \)

- \( \text{rank}(A) = r \) is minimum size vector needed to faithfully reconstruct \( y \) from \( x \)
Coding interpretation of rank

- \( \text{rank}(BC) \leq \text{rank}(B) \) because \( \text{range}(BC) \subseteq \text{range}(B) \)

- transpose implies \( \text{rank}(BC) \leq \text{rank}(C) \)

- factorization converse comes from QR
Application: fast matrix-vector multiplication

- need to compute matrix-vector product $y = Ax$, $A \in \mathbb{R}^{m \times n}$
- $A$ has known factorization $A = BC$, $B \in \mathbb{R}^{m \times r}$
- computing $y = Ax$ directly: $mn$ operations
- computing $y = Ax$ as $y = B(Cx)$ (compute $z = Cx$ first, then $y = Bz$): $rn + mr = (m + n)r$ operations
- savings can be considerable if $r \ll \min\{m, n\}$
Full rank matrices

for $A \in \mathbb{R}^{m \times n}$ we always have $\text{rank}(A) \leq \min(m, n)$

we say $A$ is full rank if $\text{rank}(A) = \min(m, n)$

- for square matrices, full rank means nonsingular
- for skinny matrices ($m \geq n$), full rank means columns are independent
- for fat matrices ($m \leq n$), full rank means rows are independent