Rank

Rank of a matrix

we define the *rank* of $A \in \mathbb{R}^{m \times n}$ as

$$\boxed{\textbf{Rank}(A) = \textbf{dim}\,\textbf{range}(A)}$$

(nontrivial) **facts:**

- $\textbf{Rank}(A) = \textbf{Rank}(A^\mathsf{T})$
- $\textbf{Rank}(A)$ is maximum number of independent columns (or rows) of A hence $\textbf{Rank}(A) \leq \textbf{min}(m, n)$

Conservation of dimension

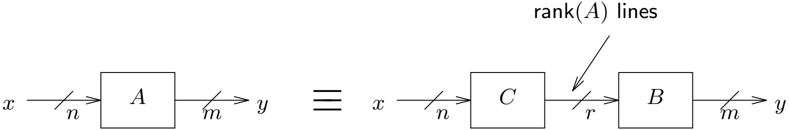
$$\textbf{dim}\,\textbf{range}(A) + \textbf{dim}\,\textbf{null}(A) = n$$

- **Rank**(A) is dimension of set 'hit' by the mapping $y = Ax$
- **dim null**(A) is dimension of set of x 'crushed' to zero by $y = Ax$
- 'conservation of dimension': each dimension of input is either crushed to zero or ends up in output
- roughly speaking:
  - n is number of degrees of freedom in input x
  - **dim null**(A) is number of degrees of freedom lost in the mapping from x to $y = Ax$
  - **Rank**(A) is number of degrees of freedom in output y

'Coding' interpretation of rank

$$\textbf{Rank}(BC) \leq \textbf{min}\{\textbf{Rank}(B), \textbf{Rank}(C)\}$$

- hence if $A = BC$ with $B \in \mathbb{R}^{m \times r}$, $C \in \mathbb{R}^{r \times n}$, then $\textbf{Rank}(A) \leq r$
- conversely: if $\textbf{rank}(A) = r$ then $A \in \mathbb{R}^{m \times n}$ can be factored as $A = BC$ with $B \in \mathbb{R}^{m \times r}$, $C \in \mathbb{R}^{r \times n}$:
- $\textbf{rank}(A) = r$ is minimum size vector needed to faithfully reconstruct y from x



Application: fast matrix-vector multiplication

- need to compute matrix-vector product $y = Ax$, $A \in \mathbb{R}^{m \times n}$
- A has known factorization $A = BC$, $B \in \mathbb{R}^{m \times r}$
- computing $y = Ax$ directly: mn operations
- computing $y = Ax$ as $y = B(Cx)$ (compute $z = Cx$ first, then $y = Bz$): $rn + mr = (m + n)r$ operations
- savings can be considerable if $r \ll \mathbf{min}\{m, n\}$

Full rank matrices

for $A \in \mathbb{R}^{m \times n}$ we always have $\textbf{Rank}(A) \leq \textbf{min}(m, n)$

we say A is *full rank* if $\textbf{Rank}(A) = \textbf{min}(m, n)$

- for **square** matrices, full rank means nonsingular
- for **skinny** matrices $(m \geq n)$, full rank means columns are independent
- for **fat** matrices $(m \leq n)$, full rank means rows are independent