Range and Null Space

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Nullspace of a matrix

the *nullspace* of $A \in \mathbb{R}^{m \times n}$ is defined as

$$\textbf{null}(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

- $\textbf{null}(A)$ is set of vectors mapped to zero by $y = Ax$
- $\textbf{null}(A)$ is set of vectors orthogonal to all rows of A

$\textbf{null}(A)$ gives *ambiguity* in x given $y = Ax$:

- if $y = Ax$ and $z \in \textbf{null}(A)$, then $y = A(x + z)$
- conversely, if $y = Ax$ and $y = A\tilde{x}$, then $\tilde{x} = x + z$ for some $z \in \textbf{null}(A)$

$\textbf{null}(A)$ is also written $\mathcal{N}(A)$

Zero nullspace

A is called *one-to-one* if 0 is the only element of its nullspace

$$\textbf{null}(A) = \{0\}$$

Equivalently,

- x can always be uniquely determined from $y = Ax$ ($i.e.$, the linear transformation $y = Ax$ doesn't 'lose' information)
- mapping from x to Ax is one-to-one: different x's map to different y's
- columns of A are independent (hence, a basis for their span)
- A has a *left inverse*, $i.e.$, there is a matrix $B \in \mathbb{R}^{n \times m}$ s.t. $BA = I$
- $A^\mathsf{T} A$ is invertible

Zero nullspace

- if A has a left inverse then $\textbf{null}(A) = \{0\}$ (proof by contradiction)
- $\textbf{null}(A) = \textbf{null}(A^\mathsf{T} A)$
- if $\textbf{null}(A) = \{0\}$ then A is left invertible, because $A^\mathsf{T} A$ is invertible, so $B = (A^\mathsf{T} A)^{-1} A^\mathsf{T}$ is a left inverse

Two interpretations of nullspace

suppose $z \in$ **null**(A), and $y = Ax$ represents *measurement* of x

- z is undetectable from sensors — get zero sensor readings
- x and $x + z$ are indistinguishable from sensors: $Ax = A(x + z)$

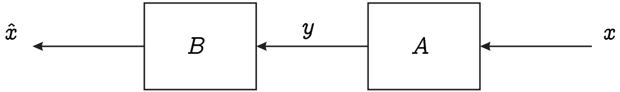
null(A) characterizes *ambiguity* in x from measurement $y = Ax$

alternatively, if $y = Ax$ represents *output* resulting from input x

- z is an input with no result
- x and $x + z$ have same result

null(A) characterizes *freedom of input choice* for given result

Left invertibility and estimation



- apply left-inverse B at output of A
- then estimate $\hat{x} = BAx = x$ as desired
- *non-unique:* both B and C are left inverses of A

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

Range of a matrix

the *range* of $A \in \mathbb{R}^{m \times n}$ is defined as

$$\textbf{range}(A) = \{Ax \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

range(A) can be interpreted as

- the set of vectors that can be 'hit' by linear mapping $y = Ax$
- the span of columns of A
- the set of vectors y for which $Ax = y$ has a solution

range(A) is also written $\mathcal{R}(A)$

Onto matrices

A is called *onto* if $\textbf{range}(A) = \mathbb{R}^m$

equivalently,

- $Ax = y$ can be solved in x for any y
- columns of A span $\mathbb{R}^m$
- A has a *right inverse*, *i.e.*, there is a matrix $B \in \mathbb{R}^{n \times m}$ s.t. $AB = I$
- rows of A are independent
- $\textbf{null}(A^\mathsf{T}) = \{0\}$
- AA^T is invertible

Onto matrices

- if **range**$(A) = \mathbb{R}^m$ then A is right invertible. To see this, let b_i be such that $Ab_i = e_i$, and let $B = \left[\, b_1, \ldots, b_m \,\right]$, then $AB = I$.
- if A is right invertible, then **range** $A = \mathbb{R}^m$, because **range**$(A) \supset$ **range**(AB)
- A is left invertible iff A^T is right invertible

Interpretations of range

suppose $v \in$ **range**(A),$w \notin$ **range**(A)

$y = Ax$ represents *measurement* of x

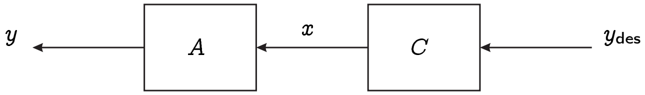
- $y = v$ is a *possible* or *consistent* sensor signal
- $y = w$ is *impossible* or *inconsistent*; sensors have failed or model is wrong

$y = Ax$ represents *output* resulting from input x

- v is a possible result or output
- w cannot be a result or output

range(A) characterizes the *possible results* or *achievable outputs*

Right invertibility and control



- apply right-inverse C at *input* of A
- then output $y = ACy_{\text{des}} = y_{\text{des}}$ as desired