Range and Null Space

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Nullspace of a matrix

the *nullspace* of $A \in \mathbb{R}^{m \times n}$ is defined as

$$\text{null}(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

- $\text{null}(A)$ is set of vectors mapped to zero by $y = Ax$
- $\text{null}(A)$ is set of vectors orthogonal to all rows of $A$

$\text{null}(A)$ gives *ambiguity* in $x$ given $y = Ax$:

- if $y = Ax$ and $z \in \text{null}(A)$, then $y = A(x + z)$
- conversely, if $y = Ax$ and $y = A\tilde{x}$, then $\tilde{x} = x + z$ for some $z \in \text{null}(A)$

$\text{null}(A)$ is also written $\mathcal{N}(A)$
Zero nullspace

\( A \) is called **one-to-one** if 0 is the only element of its nullspace

\[ \text{null}(A) = \{0\} \]

Equivalently,

- \( x \) can always be uniquely determined from \( y = Ax \)  
  *(i.e., the linear transformation \( y = Ax \) doesn't 'lose' information)*
- mapping from \( x \) to \( Ax \) is one-to-one: different \( x \)'s map to different \( y \)'s
- columns of \( A \) are independent (hence, a basis for their span)
- \( A \) has a **left inverse**, i.e., there is a matrix \( B \in \mathbb{R}^{n \times m} \) s.t. \( BA = I \)
- \( A^T A \) is invertible
If $A$ has a left inverse then $\text{null}(A) = \{0\}$ (proof by contradiction).

$\text{null}(A) = \text{null}(A^T A)$

If $\text{null}(A) = \{0\}$ then $A$ is left invertible, because $A^T A$ is invertible, so $B = (A^T A)^{-1} A^T$ is a left inverse.
**Two interpretations of nullspace**

suppose \( z \in \text{null}(A) \), and \( y = Ax \) represents *measurement* of \( x \)

- \( z \) is undetectable from sensors — get zero sensor readings
- \( x \) and \( x + z \) are indistinguishable from sensors: \( Ax = A(x + z) \)

\( \text{null}(A) \) characterizes *ambiguity* in \( x \) from measurement \( y = Ax \)

alternatively, if \( y = Ax \) represents *output* resulting from input \( x \)

- \( z \) is an input with no result
- \( x \) and \( x + z \) have same result

\( \text{null}(A) \) characterizes *freedom of input choice* for given result
Left invertibility and estimation

- apply left-inverse $B$ at output of $A$
- then estimate $\hat{x} = BAx = x$ as desired
- *non-unique:* both $B$ and $C$ are left inverses of $A$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$
Range of a matrix

the *range* of $A \in \mathbb{R}^{m \times n}$ is defined as

$$\text{range}(A) = \{Ax \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

*range*(A) can be interpreted as

- the set of vectors that can be ‘hit’ by linear mapping $y = Ax$
- the span of columns of $A$
- the set of vectors $y$ for which $Ax = y$ has a solution

*range*(A) is also written $\mathcal{R}(A)$
Onto matrices

$A$ is called onto if $\text{range}(A) = \mathbb{R}^m$ equivalently, 

- $Ax = y$ can be solved in $x$ for any $y$
- columns of $A$ span $\mathbb{R}^m$
- $A$ has a right inverse, i.e., there is a matrix $B \in \mathbb{R}^{n \times m}$ s.t. $AB = I$
- rows of $A$ are independent
- $\text{null}(A^T) = \{0\}$
- $AA^T$ is invertible
Onto matrices

- if \( \text{range}(A) = \mathbb{R}^m \) then \( A \) is right invertible. To see this, let \( b_i \) be such that \( Ab_i = e_i \), and let \( B = [b_1, \ldots, b_m] \), then \( AB = I \).

- if \( A \) is right invertible, then \( \text{range} \, A = \mathbb{R}^m \), because \( \text{range}(A) \supseteq \text{range}(AB) \)

- \( A \) is left invertible iff \( A^\top \) is right invertible
Interpretations of range

Suppose $v \in \text{range}(A), w \not\in \text{range}(A)$

$y = Ax$ represents \textit{measurement} of $x$

- $y = v$ is a \textit{possible} or \textit{consistent} sensor signal
- $y = w$ is \textit{impossible} or \textit{inconsistent}; sensors have failed or model is wrong

$y = Ax$ represents \textit{output} resulting from input $x$

- $v$ is a possible result or output
- $w$ cannot be a result or output

$\text{range}(A)$ characterizes the \textit{possible results} or \textit{achievable outputs}
Right invertibility and control

- apply right-inverse $C$ at *input* of $A$
- then output $y = ACy_{\text{des}} = y_{\text{des}}$ as desired