Orthogonality
Inner product

\[ \langle x, y \rangle := x_1y_1 + x_2y_2 + \cdots + x_ny_n = x^T y \]

important properties:

- \( \langle \alpha x, y \rangle = \alpha \langle x, y \rangle \)
- \( \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \)
- \( \langle x, y \rangle = \langle y, x \rangle \)
- \( \langle x, x \rangle \geq 0 \)
- \( \langle x, x \rangle = 0 \iff x = 0 \)

\( f(y) = \langle x, y \rangle \) is linear function : \( \mathbb{R}^n \to \mathbb{R} \), with linear map defined by row vector \( x^T \)
Cauchy-Schwarz inequality and angle between vectors

for any $x, y \in \mathbb{R}^n$

\[ |x^T y| \leq \|x\|\|y\| \]

► (unsigned) angle between vectors in $\mathbb{R}^n$ defined as

\[ \theta = \angle(x, y) = \cos^{-1}\left(\frac{x^T y}{\|x\|\|y\|}\right) \]

► thus $x^T y = \|x\|\|y\| \cos \theta$
Special cases

- **x and y are aligned**: $\theta = 0$; $x^T y = ||x|| ||y||$
  (if $x \neq 0$) $y = \alpha x$ for some $\alpha \geq 0$

- **x and y are opposed**: $\theta = \pi$; $x^T y = -||x|| ||y||$
  (if $x \neq 0$) $y = -\alpha x$ for some $\alpha \geq 0$

- **x and y are orthogonal**: $\theta = \pi/2$ or $-\pi/2$; $x^T y = 0$
  denoted $x \perp y$
**Angles**

Interpretation of $x^T y > 0$ and $x^T y < 0$

- $x^T y > 0$ means $\angle(x, y)$ is acute
- $x^T y < 0$ means $\angle(x, y)$ is obtuse
Halfspaces

A halfspace with outward normal vector $y$, and boundary passing through 0

$$H = \{x \mid x^T y \leq 0\}$$
for any set $S \subset \mathbb{R}^n$, the **orthogonal complement** is

$$S^\perp = \{ x \mid x^T y = 0 \text{ for all } y \in S \}$$

- $S^\perp$ is always a subspace
- $S^\perp$ is the set of all vectors $x$, each of which is orthogonal to every vector in $S$
Orthonormal set of vectors

set of vectors \( \{ u_1, \ldots, u_k \} \subset \mathbb{R}^n \) is

- **normalized** if \( \| u_i \| = 1 \), \( i = 1, \ldots, k \)
  
  (\( u_i \) are called *unit vectors* or *direction vectors*)

- **orthogonal** if \( u_i \perp u_j \) for \( i \neq j \)

- **orthonormal** if both

**slang:** we say ‘\( u_1, \ldots, u_k \) are orthonormal vectors’ but orthonormality (like independence) is a property of a *set* of vectors, not vectors individually

in terms of \( U = [ u_1 \cdots u_k ] \), orthonormal means

\[
U^T U = I_k
\]
Orthonormality

an orthonormal set of vectors is independent

- to see this, multiply $Ux = 0$ by $U^T$
- hence $\{u_1, \ldots, u_k\}$ is an orthonormal basis for $\text{span}(u_1, \ldots, u_k) = \text{range}(U)$

- warning: if $k < n$ then $UU^T \neq I$ (since its rank is at most $k$)
  (more on this matrix later . . .)
Orthonormal basis for $\mathbb{R}^n$

A matrix $U$ is called *orthogonal* if

$$U \text{ is square and } U^T U = I$$

- the set of columns $u_1, \ldots, u_n$ is an orthonormal *basis* for $\mathbb{R}^n$
- (you’d think such matrices would be called *orthonormal*, not *orthogonal*)
- it follows that $U^{-1} = U^T$, and hence also $UU^T = I$, *i.e.*, 
  $$\sum_{i=1}^{n} u_i u_i^T = I$$
Expansion in orthonormal basis

suppose $U$ is orthogonal, so $x = UU^T x$, i.e.,

$$x = \sum_{i=1}^{n} (u_i^T x) u_i$$

- $u_i^T x$ is called the component of $x$ in the direction $u_i$
- $a = U^T x$ resolves $x$ into the vector of its $u_i$ components
- $x = Ua$ reconstitutes $x$ from its $u_i$ components
- $x = Ua = \sum_{i=1}^{n} a_i u_i$ is called the (u_i-) expansion of $x$
Geometric interpretation

if $U$ has orthonormal columns then transformation $w = Uz$

- preserves *norm* of vectors, i.e., $\|Uz\| = \|z\|$
- preserves *angles* between vectors, i.e., $\angle(Uz, U\tilde{z}) = \angle(z, \tilde{z})$
- we say $U$ is *isometric*, it preserves distances
Example: Rotation

rotation by $\theta$ in $\mathbb{R}^2$ is given by

$$y = U_\theta x, \quad U_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

since $e_1 \mapsto (\cos \theta, \sin \theta), \ e_2 \mapsto (-\sin \theta, \cos \theta)$
Example: Reflection

reflection across line $x_2 = x_1 \tan(\theta/2)$ is given by

$$y = R_\theta x, \quad R_\theta = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$$

since $e_1 \rightarrow (\cos \theta, \sin \theta), \ e_2 \rightarrow (\sin \theta, -\cos \theta)$