Multi-objective least-squares

- multi-objective least-squares
- regularized least-squares

Multi-objective least-squares

in many problems we have two (or more) objectives

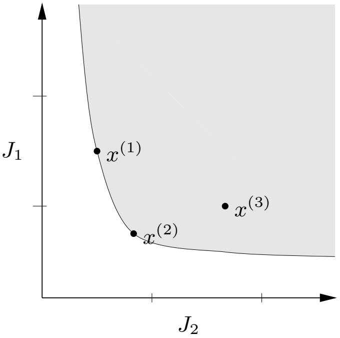
- we want $J_1 = ||Ax - y||^2$ small
- and also $J_2 = ||Fx - g||^2$ small

($x \in \mathbb{R}^n$ is the variable)

- usually the objectives are *competing*
- we can make one smaller, at the expense of making the other larger

common example: $F = I$, $g = 0$; we want $||Ax - y||$ small, with small x

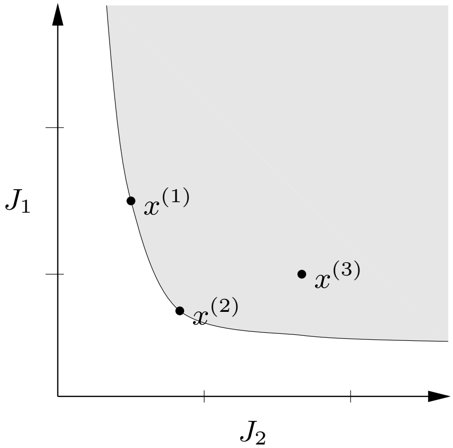
Plot of achievable objective pairs



- plot (J_2, J_1) for every x
- $x \in \mathbb{R}^n$, but this plot is in $\mathbb{R}^2$
- point labeled $x^{(1)}$ is really $\left(J_2(x^{(1)}), J_1(x^{(1)})\right)$

Optimal trade-off curve

- shaded area shows (J_2, J_1) achieved by some $x \in \mathbb{R}^n$
- clear area shows (J_2, J_1) not achieved by any $x \in \mathbb{R}^n$
- boundary of region is called *optimal trade-off curve*
- corresponding x are called *Pareto optimal* for the two objectives $||Ax - y||^2$, $||Fx - g||^2$



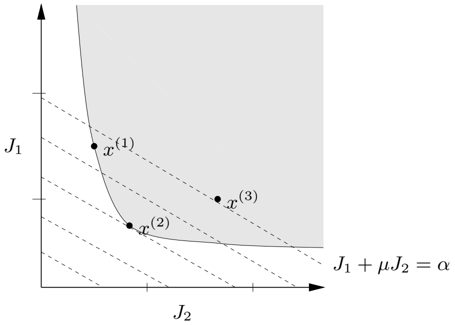
three example choices of x: $x^{(1)}$, $x^{(2)}$, $x^{(3)}$

- $x^{(3)}$ is worse than $x^{(2)}$ on both counts (J_2 and J_1)
- $x^{(1)}$ is better than $x^{(2)}$ in J_2, but worse in J_1

Weighted-sum objective

- to find Pareto optimal points,
  $i.e.$, x's on optimal trade-off curve,
  we minimize *weighted-sum objective*

$$J_1 + \mu J_2 = \|Ax - y\|^2 + \mu\|Fx - g\|^2$$



- parameter $\mu \geq 0$ gives relative weight between J_1 and J_2
- points where weighted sum is constant, $J_1 + \mu J_2 = \alpha$, correspond to line with slope $-\mu$ on (J_2, J_1) plot
- $x^{(2)}$ minimizes weighted-sum objective for μ shown
- by varying μ from 0 to $+\infty$, can sweep out entire *optimal tradeoff curve*

Minimizing weighted-sum objective

can express weighted-sum objective as ordinary least-squares objective:

$$\begin{aligned} \|Ax - y\|^2 + \mu\|Fx - g\|^2 &= \left\| \begin{bmatrix} A \\ \sqrt{\mu}F \end{bmatrix} x - \begin{bmatrix} y \\ \sqrt{\mu}g \end{bmatrix} \right\|^2 \\ &= \|\tilde{A}x - \tilde{y}\|^2 \end{aligned}$$

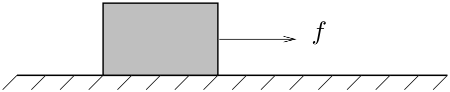
where

$$\tilde{A} = \begin{bmatrix} A \\ \sqrt{\mu}F \end{bmatrix}, \qquad \tilde{y} = \begin{bmatrix} y \\ \sqrt{\mu}g \end{bmatrix}$$

hence solution is (assuming $\tilde{A}$ full rank)

$$\begin{aligned} x &= \left(\tilde{A}^T\tilde{A}\right)^{-1}\tilde{A}^T\tilde{y} \\ &= \left(A^TA + \mu F^TF\right)^{-1}\left(A^Ty + \mu F^Tg\right) \end{aligned}$$

Example



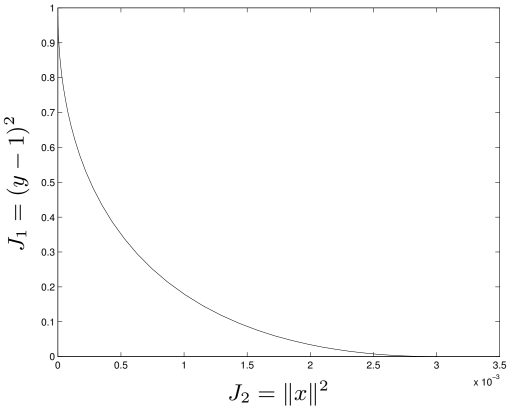
- unit mass at rest subject to forces x_i for $i-1 < t \leq i$, $i = 1, \ldots, 10$
- $y \in \mathbb{R}$ is position at $t = 10$; $y = a^\top x$ where $a \in \mathbb{R}^{10}$
- $J_1 = (y-1)^2$ (final position error squared)
- $J_2 = ||x||^2$ (sum of squares of forces)

weighted-sum objective: $(a^\top x - 1)^2 + \mu ||x||^2$

optimal x:

$$x = \left(aa^\top + \mu I\right)^{-1} a$$

Optimal trade-off curve



- upper left corner of optimal trade-off curve corresponds to $x = 0$
- bottom right corresponds to input that yields $y = 1$, $i.e.$, $J_1 = 0$

Regularized least-squares

when $F = I$, $g = 0$ the objectives are

$$J_1 = \|Ax - y\|^2, \qquad J_2 = \|x\|^2$$

minimizer of weighted-sum objective,

$$x = \left(A^\mathsf{T} A + \mu I\right)^{-1} A^\mathsf{T} y,$$

is called *regularized* least-squares (approximate) solution of $Ax \approx y$

- also called *Tychonov regularization*
- for $\mu > 0$, works for *any* A (no restrictions on shape, rank ...)

estimation/inversion application:

- $Ax - y$ is sensor residual
- prior information: x small
- regularized solution trades off sensor fit, size of x