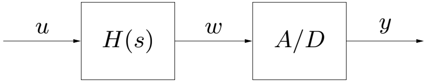
Example: Least-squares filtering

Example: estimation / filtering



- signal u is piecewise constant, period $1\,\mathrm{sec}$, $0 \leq t \leq 10$:

$$u(t) = x_j, \quad j-1 \leq t < j, \quad j = 1, \ldots, 10$$

- filtered by system with impulse response $h(t)$:

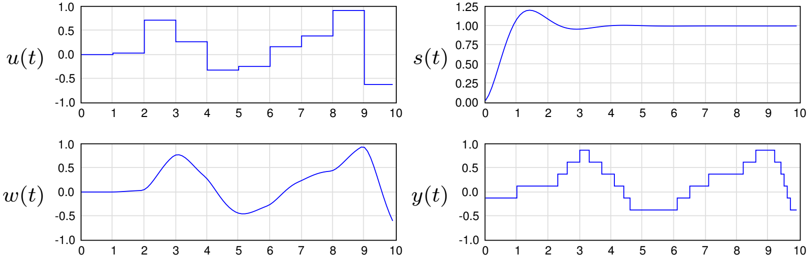
$$w(t) = \int_0^t h(t-\tau) u(\tau)\, d\tau$$

- sample at 10Hz: $\tilde{y}_i = w(0.1i)$, $i = 1, \ldots, 100$
- 3-bit quantization: $y_i = Q(\tilde{y}_i)$, $i = 1, \ldots, 100$, where Q is 3-bit quantizer characteristic

$$Q(a) = (1/4)\left(\mathbf{round}(4a + 1/2) - 1/2\right)$$

- **problem:** estimate $x \in \mathbb{R}^{10}$ given $y \in \mathbb{R}^{100}$

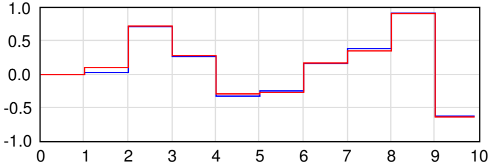
Linear model



we have $y = Ax + v$, where

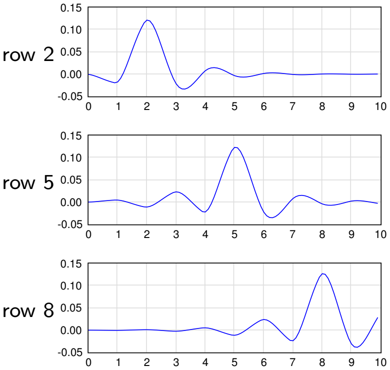
- $A \in \mathbb{R}^{100\times 10}$ is given by $A_{ij} = \displaystyle\int_{j-1}^{j} h(0.1i - \tau)\, d\tau$
- $v \in \mathbb{R}^{100}$ is *quantization error:* $v_i = Q(\tilde{y}_i) - \tilde{y}_i$ (so $|v_i| \le 0.125$)

Results from least-squares estimation



- plot shows *least-squares estimate:* $x_{\text{ls}} = (A^\mathsf{T} A)^{-1} A^\mathsf{T} y$ in red
- RMS error is $\dfrac{\|x - x_{\text{ls}}\|}{\sqrt{10}} = 0.03$
- *better* than if we had no filtering! (RMS error 0.07)

Rows of the left-inverse



- some rows of $B_{\mathsf{ls}} = (A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}$
- rows show how sampled measurements of y are used to form estimate of x_i for $i = 2, 5, 8$
- to estimate x_5, which is the original input signal for $4 \leq t < 5$, we mostly use $y(t)$ for $3 \leq t \leq 7$