Linear dynamical systems with inputs & outputs

- inputs & outputs: interpretations
- impulse and step responses
- examples
Inputs & outputs

recall continuous-time time-invariant LDS has form

\[ \dot{x} = Ax + Bu, \quad y = Cx + Du \]

- \(Ax\) is called the **drift term** (of \(\dot{x}\))
- \(Bu\) is called the input term (of \(\dot{x}\))

picture, with \(B \in \mathbb{R}^{2 \times 1}\):

\[ \dot{x}(t) \text{ (with } u(t) = 1) \]
\[ \dot{x}(t) \text{ (with } u(t) = -1.5) \]
Interpretations

write \( \dot{x} = Ax + b_1 u_1 + \cdots + b_m u_m \), where \( B = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix} \)

- state derivative is sum of autonomous term \((Ax)\) and one term per input \((b_i u_i)\)
- each input \(u_i\) gives another degree of freedom for \(\dot{x}\) (assuming columns of \(B\) independent)

write \( \dot{x} = Ax + Bu \) as \( \dot{x}_i = \tilde{a}_i^T x + \tilde{b}_i^T u \), where \( \tilde{a}_i^T, \tilde{b}_i^T \) are the rows of \(A, B\)

- \(i\)th state derivative is linear function of state \(x\) and input \(u\)
Response to input

- The solution to $\dot{x} = Ax + Bu$ is

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A} Bu(\tau) \, d\tau$$

- $e^{tA}x(0)$ is the unforced or autonomous response

- $e^{tA}B$ is called the input-to-state impulse response or impulse matrix
Impulse response

impulse response $h(t) = Ce^{tA}B + D\delta(t)$

with $x(0) = 0$, $y = h \ast u$, i.e.,

$$y_i(t) = \sum_{j=1}^{m} \int_{0}^{t} h_{ij}(t - \tau)u_j(\tau) \, d\tau$$

interpretations:

- $h_{ij}(t)$ is impulse response from $j$th input to $i$th output
- $h_{ij}(t)$ gives $y_i(t)$ when $u(t) = e_j \delta(t)$
- $h_{ij}(\tau)$ shows how dependent output $i$ is, on what input $j$ was, $\tau$ seconds ago
- $i$ indexes output; $j$ indexes input; $\tau$ indexes time lag
Mass-spring example

- unit masses, springs, dampers
- $u_1$ is tension between 1st & 2nd masses
- $u_2$ is tension between 2nd & 3rd masses
- $y \in \mathbb{R}^3$ is displacement of masses 1, 2, 3

\[
x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}
\]
Mass-spring example

The system is:

\[
\dot{x} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-2 & 1 & 0 & -2 & 1 & 0 \\
1 & -2 & 1 & 1 & -2 & 1 \\
0 & 1 & -2 & 0 & 1 & -2
\end{bmatrix} x + \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
1 & 0 \\
0 & -1
\end{bmatrix} u + \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]

eigenvalues of \( A \) are

\[-1.71 \pm i0.71, \quad -1.00 \pm i1.00, \quad -0.29 \pm i0.71\]
Example: Impulse response

roughly speaking:

- impulse at \( u_1 \) affects third mass less than other two
- impulse at \( u_2 \) affects first mass later than other two
Discretization with piecewise constant inputs

linear system \( \dot{x} = Ax + Bu, \ y = Cx + Du \)

suppose \( u_d : \mathbb{Z}_+ \to \mathbb{R}^m \) is a sequence, and
\[
\begin{align*}
  u(t) &= u_d(k) & \text{for } & kh \leq t < (k + 1)h, & k = 0, 1, \ldots
\end{align*}
\]

define sequences
\[
\begin{align*}
  x_d(k) &= x(kh), & y_d(k) &= y(kh), & k = 0, 1, \ldots
\end{align*}
\]

\( h > 0 \) is called the sample interval (for \( x \) and \( y \)) or update interval (for \( u \))

\( u \) is piecewise constant (called zero-order-hold)

\( x_d, y_d \) are sampled versions of \( x, y \)
Discretization with piecewise constant inputs

\[
x_d(k + 1) = x((k + 1)h)
= e^{hA} x(kh) + \int_0^h e^{\tau A} B u((k + 1)h - \tau) \, d\tau
= e^{hA} x_d(k) + \left( \int_0^h e^{\tau A} \, d\tau \right) B u_d(k)
\]

\(x_d, u_d, \) and \(y_d\) satisfy discrete-time LDS equations

\[
x_d(k + 1) = A_d x_d(k) + B_d u_d(k), \quad y_d(k) = C_d x_d(k) + D_d u_d(k)
\]

where

\[
A_d = e^{hA}, \quad B_d = \left( \int_0^h e^{\tau A} \, d\tau \right) B, \quad C_d = C, \quad D_d = D
\]

called discretized system. If \(A\) is invertible, we can express integral as

\[
\int_0^h e^{\tau A} \, d\tau = A^{-1} (e^{hA} - I)
\]
Example: Force on mass

Newton’s law gives continuous-time LDS

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)
\]

\[
y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)
\]

let’s compute the discretization

\[
A_d = e^{Ah}
\]

\[
= I + Ah + \frac{1}{2} A^2 h^2 + \cdots
\]

\[
= I + Ah
\]

\[
= \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}
\]
Example: Force on mass

\[ B_d = \int_0^h e^{As} B \, ds \]

\[ = \int_0^h \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \, ds \]

\[ = \int_0^h s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \, ds = \begin{bmatrix} \frac{1}{2} h^2 \\ h \end{bmatrix} \]

so the discretization is

\[ x_d(k + 1) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} x_d(k) + \begin{bmatrix} \frac{1}{2} h^2 \\ h \end{bmatrix} u_d(k) \]

\[ y_d(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_d(k) \]
Stability of discretization

**stability**: if eigenvalues of $A$ are $\lambda_1, \ldots, \lambda_n$, then eigenvalues of $A_d$ are $e^{h\lambda_1}, \ldots, e^{h\lambda_n}$

discretization preserves stability properties since

$$\Re \lambda_i < 0 \iff |e^{h\lambda_i}| < 1$$

for $h > 0$
Extensions and variations

- **offsets**: updates for $u$ and sampling of $x$, $y$ are offset in time
- **multirate**: $u_i$ updated, $y_i$ sampled at different intervals
  (usually integer multiples of a common interval $h$)

both very common in practice
Discrete-time systems

discrete-time LDS:

\[
x(t + 1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)
\]

we have:

\[
x(1) = Ax(0) + Bu(0),
\]

\[
x(2) = Ax(1) + Bu(1)
\]

\[
= A^2 x(0) + ABu(0) + Bu(1),
\]

and in general, for \( t \in \mathbb{Z}_+ \),

\[
x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{t-1-\tau} Bu(\tau)
\]
Discrete-time systems

Solution is

\[ x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)} B u(\tau) \]

write this as

\[ y(t) = C A^t x(0) + H \ast u \]

where \( \ast \) is discrete-time convolution

\[ y(t) = C A^t x(0) + \sum_{\tau=0}^{t} H(t - \tau) u(\tau) \]

and

\[ H(t) = \begin{cases} D, & t = 0 \\ C A^{t-1} B, & t > 0 \end{cases} \]

is the impulse response
Block Toeplitz matrices

we have

\[
\begin{bmatrix}
y(0) \\
y(1) \\
y(2) \\
\vdots \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
D & & & \\
CB & D & & \\
CAB & CB & D & \\
& \ddots & \ddots & \ddots \\
CA^{t-1}B & CA^{t-2}B & \cdots & CB & D
\end{bmatrix}
\begin{bmatrix}
u(0) \\
u(1) \\
u(2) \\
\vdots \\
u(t)
\end{bmatrix} +
\begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^t
\end{bmatrix}
x(0)
\]

- this matrix gives the output sequence \( y(0), y(1), \ldots \) in terms of the input sequence \( u(0), u(1), \ldots \) and the initial state \( x(0) \)
- **block Toeplitz** means blocks are constant along diagonals from top-left to bottom right
- we can use this to find controllers and estimators
Example: Point mass

unit point mass, with actuators applying force in directions

\[
v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

has dynamics

\[
x(k + 1) = \begin{bmatrix} 1 & h & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} \frac{1}{2} h^2 & 0 \\ h & 0 \\ 0 & \frac{1}{2} h^2 \\ 0 & h \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}
\]

\[
y(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(k)
\]

here

- \(x_1, x_2 = \) position, velocity in \(x\)-direction
- \(x_3, x_4 = \) position, velocity in \(y\)-direction
- \(h = \) sample time; we'll use \(h = 1\).
- \(u_i(k) = \) current applied to actuator \(i\) at time \(k\).
Example: Point mass

we would like to drive it through the positions

\[ y(20) = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \quad y(40) = \begin{bmatrix} 10 \\ -1 \end{bmatrix} \quad y(70) = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \]

at the above times

we have

\[ y(t) = \sum_{\tau=0}^{t-1} CA^{t-1-\tau} Bu(\tau) + Du(t) \]

this gives the rows of

\[ \begin{bmatrix} y(20) \\ y(40) \\ y(70) \end{bmatrix} = A_{\text{act}} \begin{bmatrix} u(0) \\ \vdots \\ u(70) \end{bmatrix} \]

here \( A_{\text{act}} \) is \( 6 \times 213 \).
Example: Point mass

let’s find the minimum norm sequence of forces that meets the specifications

\[
\begin{bmatrix}
  u(0) \\
  \vdots \\
  u(70)
\end{bmatrix} = \Lambda_{\text{act}}^{\dagger} \begin{bmatrix}
  5 \\
  3 \\
  10 \\
  -1 \\
  4 \\
  1
\end{bmatrix}
\]

trajectory is

![Graph showing the trajectory with specified points and a series of forces and parameters.](image-url)
Example: Point mass

sequence of force inputs is
the step response or step matrix is given by

\[ s(t) = \int_0^t h(\tau) \, d\tau \]

interpretations:

- \( s_{ij}(t) \) is step response from \( j \)th input to \( i \)th output
- \( s_{ij}(t) \) gives \( y_i \) when \( u = e_j \) for \( t \geq 0 \)

for invertible \( A \), we have

\[ s(t) = CA^{-1} \left( e^{tA} - I \right) B + D \]
Circuit example

- $u(t) \in \mathbb{R}$ is input (drive) voltage
- $x_i$ is voltage across $C_i$
- output is state: $y = x$
- unit resistors, unit capacitors
- step response matrix shows delay to each node
Circuit example

System is

\[
\dot{x} = \begin{bmatrix}
-3 & 1 & 1 & 0 \\
1 & -1 & 0 & 0 \\
1 & 0 & -2 & 1 \\
0 & 0 & 1 & -1 \\
\end{bmatrix} x + \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
\end{bmatrix} u, \quad y = x
\]

eigenvalues of \( A \) are

\[
-0.17, \quad -0.66, \quad -2.21, \quad -3.96
\]
Circuit example

step response matrix \( s(t) \in \mathbb{R}^{4 \times 1} \):

- shortest delay to \( x_1 \); longest delay to \( x_4 \)
- delays consistent with slowest (i.e., dominant) eigenvalue \(-0.17\)
DC or static gain matrix

- DC gain describes system under *static* conditions, i.e., $x$, $u$, $y$ constant:

  
  \[
  0 = \dot{x} = Ax + Bu, \quad y = Cx + Du
  \]

  eliminate $x$ to get $y = H_0 u$ where

  
  \[
  H_0 = -CA^{-1}B + D
  \]

- if system is stable,

  \[
  H_0 = \int_0^\infty h(t) \, dt = \lim_{t \to \infty} s(t)
  \]

  if $u(t) \to u_\infty \in \mathbb{R}^m$, then $y(t) \to y_\infty \in \mathbb{R}^p$ where $y_\infty = H_0 u_\infty$
DC gain matrix

DC gain matrix for spring-mass example:

\[ H(0) = \begin{bmatrix} 1/4 & 1/4 \\ -1/2 & 1/2 \\ -1/4 & -1/4 \end{bmatrix} \]

DC gain matrix for circuit example:

\[ H(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \]

(do these make sense?)
Causality

interpretation of

\[ x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau) \, d\tau \]
\[ y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau) \, d\tau + Du(t) \]

for \( t \geq 0 \):

- current state \((x(t))\) and output \((y(t))\) depend on past input \((u(\tau)\text{ for } \tau \leq t)\)
- i.e., mapping from input to state and output is causal (with fixed initial state)
Idea of state

$x(t)$ is called state of system at time $t$ since:

- future output depends only on current state and future input
- future output depends on past input only through current state
- state summarizes effect of past inputs on future output
- state is bridge between past inputs and future outputs
Change of coordinates

start with LDS $\dot{x} = Ax + Bu, \ y = Cx + Du$

change coordinates in $\mathbb{R}^n$ to $\tilde{x}$, with $x = T\tilde{x}$

then

$$\dot{\tilde{x}} = T^{-1}\dot{x} = T^{-1}(Ax + Bu) = T^{-1}AT\tilde{x} + T^{-1}Bu$$

hence LDS can be expressed as

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \quad y = \tilde{C}\tilde{x} + \tilde{D}u$$

where

$$\tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B, \quad \tilde{C} = CT, \quad \tilde{D} = D$$

TF is same (since $u, y$ aren’t affected):

$$\tilde{C} (sI - \tilde{A})^{-1} \tilde{B} + \tilde{D} = C (sI - A)^{-1}B + D$$
Standard forms for LDS

can change coordinates to put $A$ in various forms (diagonal, real modal, Jordan . . .)
e.g., to put LDS in \textit{diagonal form}, find $T$ s.t.

$$T^{-1}AT = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

write

$$T^{-1}B = \begin{bmatrix} \tilde{b}_1^T \\ \vdots \\ \tilde{b}_n^T \end{bmatrix}, \quad CT = \begin{bmatrix} \tilde{c}_1 & \cdots & \tilde{c}_n \end{bmatrix}$$