Linear dynamical systems with inputs & outputs

- inputs & outputs: interpretations
- impulse and step responses
- examples
**Inputs & outputs**

recall continuous-time time-invariant LDS has form

\[
\dot{x} = Ax + Bu, \quad y = Cx + Du
\]

- \(Ax\) is called the **drift term** (of \(\dot{x}\))
- \(Bu\) is called the input term (of \(\dot{x}\))

picture, with \(B \in \mathbb{R}^{2 \times 1}\):

\[
\dot{x}(t) \text{ (with } u(t) = 1) \quad \dot{x}(t) \text{ (with } u(t) = -1.5)
\]
Interpretations

write $\dot{x} = Ax + b_1 u_1 + \cdots + b_m u_m$, where $B = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}$

- state derivative is sum of autonomous term ($Ax$) and one term per input ($b_i u_i$)

- each input $u_i$ gives another degree of freedom for $\dot{x}$ (assuming columns of $B$ independent)

write $\dot{x} = Ax + Bu$ as $\dot{x}_i = \tilde{a}_i^T x + \tilde{b}_i^T u$, where $\tilde{a}_i^T, \tilde{b}_i^T$ are the rows of $A, B$

- $i$th state derivative is linear function of state $x$ and input $u$
Response to input

- The solution to $\dot{x} = Ax + Bu$ is
  \[ x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau) \, d\tau \]

- $e^{tA}x(0)$ is the unforced or autonomous response

- $e^{tA}B$ is called the input-to-state impulse response or impulse matrix
Impulse response

impulse response $h(t) = Ce^{tA}B + D\delta(t)$

with $x(0) = 0$, $y = h \ast u$, i.e.,

$$y_i(t) = \sum_{j=1}^{m} \int_{0}^{t} h_{ij}(t - \tau)u_j(\tau) \, d\tau$$

interpretations:

- $h_{ij}(t)$ is impulse response from $j$th input to $i$th output
- $h_{ij}(t)$ gives $y_i(t)$ when $u(t) = e_j\delta(t)$
- $h_{ij}(\tau)$ shows how dependent output $i$ is, on what input $j$ was, $\tau$ seconds ago
- $i$ indexes output; $j$ indexes input; $\tau$ indexes time lag
Mass-spring example

- unit masses, springs, dampers
- $u_1$ is tension between 1st & 2nd masses
- $u_2$ is tension between 2nd & 3rd masses
- $y \in \mathbb{R}^3$ is displacement of masses 1,2,3
- $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$
Mass-spring example

system is:

\[
\dot{x} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-2 & 1 & 0 & -2 & 1 & 0 \\
1 & -2 & 1 & 1 & -2 & 1 \\
0 & 1 & -2 & 0 & 1 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]

eigenvalues of \( A \) are

\[-1.71 \pm i0.71, \quad -1.00 \pm i1.00, \quad -0.29 \pm i0.71\]
Example: Impulse response

![Impulse response graphs for $u_1$ and $u_2$]

roughly speaking:

- impulse at $u_1$ affects third mass less than other two
- impulse at $u_2$ affects first mass later than other two
Discretization with piecewise constant inputs

linear system \( \dot{x} = Ax + Bu, \ y = Cx + Du \)

suppose \( u_d : \mathbb{Z}_+ \rightarrow \mathbb{R}^m \) is a sequence, and

\[
    u(t) = u_d(k) \quad \text{for} \quad kh \leq t < (k + 1)h, \quad k = 0, 1, \ldots
\]

define sequences

\[
    x_d(k) = x(kh), \quad y_d(k) = y(kh), \quad k = 0, 1, \ldots
\]

\( h > 0 \) is called the sample interval (for \( x \) and \( y \)) or update interval (for \( u \))

\( u \) is piecewise constant (called zero-order-hold)

\( x_d, y_d \) are sampled versions of \( x, y \)
Discretization with piecewise constant inputs

\[ x_d(k+1) = x((k+1)h) = e^{hA}x(kh) + \int_0^h e^{\tau A}Bu((k+1)h - \tau)\,d\tau = e^{hA}x_d(k) + \left(\int_0^h e^{\tau A}\,d\tau\right)B\,u_d(k) \]

\( x_d, u_d, \) and \( y_d \) satisfy discrete-time LDS equations

\[ x_d(k+1) = A_dx_d(k) + B_du_d(k), \quad y_d(k) = C_dx_d(k) + D_du_d(k) \]

where

\[ A_d = e^{hA}, \quad B_d = \left(\int_0^h e^{\tau A}\,d\tau\right)B, \quad C_d = C, \quad D_d = D \]

called discretized system. If \( A \) is invertible, we can express integral as

\[ \int_0^h e^{\tau A}\,d\tau = A^{-1}\left(e^{hA} - I\right) \]
Newton’s law gives continuous-time LDS

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)
\]
\[
y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)
\]

let’s compute the discretization

\[
A_d = e^{Ah}
\]
\[
= I + Ah + \frac{1}{2} A^2 h^2 + \cdots
\]
\[
= I + Ah
\]
\[
= \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}
\]
Example: Force on mass

\[ B_d = \int_0^h e^{As} B \, ds \]

\[ = \int_0^h \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \, ds \]

\[ = \int_0^h \begin{bmatrix} s \\ 1 \end{bmatrix} \, ds = \left[ \frac{1}{2} h^2 \right] \]

so the discretization is

\[ x_d(k + 1) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} x_d(k) + \begin{bmatrix} \frac{1}{2} h^2 \\ h \end{bmatrix} u_d(k) \]

\[ y_d(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_d(k) \]
Stability of discretization

**stability:** if eigenvalues of $A$ are $\lambda_1, \ldots, \lambda_n$, then eigenvalues of $A_d$ are $e^{h\lambda_1}, \ldots, e^{h\lambda_n}$

discretization preserves stability properties since

$$\Re{\lambda_i} < 0 \iff |e^{h\lambda_i}| < 1$$

for $h > 0$
Extensions and variations

- **offsets**: updates for $u$ and sampling of $x, y$ are offset in time

- **multirate**: $u_i$ updated, $y_i$ sampled at different intervals
  (usually integer multiples of a common interval $h$)

both very common in practice
Discrete-time systems

discrete-time LDS:

\[ x(t + 1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) \]

we have:

\[ x(1) = Ax(0) + Bu(0), \]

\[ x(2) = Ax(1) + Bu(1) \]
\[ = A^2 x(0) + ABu(0) + Bu(1), \]

and in general, for \( t \in \mathbb{Z}_+ \),

\[ x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)} Bu(\tau) \]
Discrete-time systems

Solution is

\[ x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)} Bu(\tau) \]

write this as

\[ y(t) = CA^t x(0) + H \ast u \]

where \( \ast \) is discrete-time convolution

\[ y(t) = CA^t x(0) + \sum_{\tau=0}^{t} H(t - \tau)u(\tau) \]

and

\[ H(t) = \begin{cases} D, & t = 0 \\ CA^{t-1} B, & t > 0 \end{cases} \]

is the impulse response
Block Toeplitz matrices

we have

\[
\begin{bmatrix}
  y(0) \\
  y(1) \\
  y(2) \\
  \vdots \\
  y(t)
\end{bmatrix}
= 
\begin{bmatrix}
  D & & & & \\
  CB & D & & & \\
  CAB & CB & D & & \\
  & \ddots & \ddots & \ddots & \\
  & & CA^{t-1}B & CA^{t-2}B & \ldots & CB & D
\end{bmatrix}
\begin{bmatrix}
  u(0) \\
  u(1) \\
  u(2) \\
  \vdots \\
  u(t)
\end{bmatrix}
+ 
\begin{bmatrix}
  C \\
  CA \\
  CA^2 \\
  \vdots \\
  CA^t
\end{bmatrix}
\begin{bmatrix}
  x(0)
\end{bmatrix}
\]

- this matrix gives the output sequence \( y(0), y(1), \ldots \) in terms of the input sequence \( u(0), u(1), \ldots \) and the initial state \( x(0) \)

- block Toeplitz means blocks are constant along diagonals from top-left to bottom right

- we can use this to find controllers and estimators
Example: Point mass

unit point mass, with actuators applying force in directions

\[ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

has dynamics

\[
x(k+1) = \begin{bmatrix} 1 & h & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} \frac{1}{2} h^2 & 0 \\ h & 0 \\ 0 & \frac{1}{2} h^2 \\ 0 & h \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}
\]

\[
y(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(k)
\]

here

- \( x_1, x_2 = \) position, velocity in \( x \)-direction
- \( x_3, x_4 = \) position, velocity in \( y \)-direction
- \( h = \) sample time; we’ll use \( h = 1 \).
- \( u_i(k) = \) current applied to actuator \( i \) at time \( k \)
Example: Point mass

we would like to drive it through the positions

\[ y(20) = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \quad y(40) = \begin{bmatrix} 10 \\ -1 \end{bmatrix} \quad y(70) = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \]

at the above times

we have

\[
y(t) = \sum_{\tau=0}^{t-1} CA^{t-1-\tau} Bu(\tau) + Du(t)
\]

this gives the rows of

\[
\begin{bmatrix} y(20) \\ y(40) \\ y(70) \end{bmatrix} = A_{\text{act}} \begin{bmatrix} u(0) \\ \vdots \\ u(70) \end{bmatrix}
\]

here \( A_{\text{act}} \) is \( 6 \times 213 \).
Example: Point mass

let's find the minimum norm sequence of forces that meets the specifications

$$\begin{bmatrix}
    u(0) \\
    \vdots \\
    u(70)
\end{bmatrix} = A_{act}^\dagger \begin{bmatrix}
    5 \\
    3 \\
    10 \\
    -1 \\
    4 \\
    1
\end{bmatrix}$$

trajectory is
Example: Point mass

sequence of force inputs is
Step response

the step response or step matrix is given by

\[ s(t) = \int_0^t h(\tau) \, d\tau \]

interpretations:

- \( s_{ij}(t) \) is step response from \( j \)th input to \( i \)th output
- \( s_{ij}(t) \) gives \( y_i \) when \( u = e_j \) for \( t \geq 0 \)

for invertible \( A \), we have

\[ s(t) = CA^{-1} \left( e^{tA} - I \right) B + D \]
Circuit example

$u(t) \in \mathbb{R}$ is input (drive) voltage

$x_i$ is voltage across $C_i$

output is state: $y = x$

unit resistors, unit capacitors

step response matrix shows delay to each node
Circuit example

The system is

\[
\dot{x} = \begin{bmatrix}
-3 & 1 & 1 & 0 \\
1 & -1 & 0 & 0 \\
1 & 0 & -2 & 1 \\
0 & 0 & 1 & -1 \\
\end{bmatrix} x + \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
\end{bmatrix} u, \quad y = x
\]

eigenvalues of \( A \) are

\[-0.17, \quad -0.66, \quad -2.21, \quad -3.96\]
Circuit example

step response matrix $s(t) \in \mathbb{R}^{4 \times 1}$:

- shortest delay to $x_1$; longest delay to $x_4$
- delays consistent with slowest (i.e., dominant) eigenvalue $-0.17$
**DC or static gain matrix**

- DC gain describes system under *static* conditions, *i.e.*, $x$, $u$, $y$ constant:

\[
0 = \dot{x} = Ax + Bu, \quad y = Cx + Du
\]

eliminate $x$ to get $y = H_0u$ where

\[
H_0 = -CA^{-1}B + D
\]

- if system is stable,

\[
H_0 = \int_0^\infty h(t) \, dt = \lim_{t \to \infty} s(t)
\]

if $u(t) \to u_\infty \in \mathbb{R}^m$, then $y(t) \to y_\infty \in \mathbb{R}^p$ where $y_\infty = H_0u_\infty$
DC gain matrix

DC gain matrix for spring-mass example:

\[
H(0) = \begin{bmatrix}
\frac{1}{4} & \frac{1}{4} \\
-\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{4} & -\frac{1}{4}
\end{bmatrix}
\]

DC gain matrix for circuit example:

\[
H(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\]

(do these make sense?)
Causality

interpretation of

\[ x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau)\,d\tau \]

\[ y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau)\,d\tau + Du(t) \]

for \( t \geq 0 \):

*current* state \((x(t))\) and output \((y(t))\) depend on *past* input \((u(\tau)\) for \( \tau \leq t \))

*i.e.*, mapping from input to state and output is *causal* (with fixed *initial* state)
**Idea of state**

\[ x(t) \] is called *state* of system at time \( t \) since:

- future output depends only on current state and future input
- future output depends on past input only through current state
- state summarizes effect of past inputs on future output
- state is bridge between past inputs and future outputs
Change of coordinates

start with LDS $\dot{x} = Ax + Bu, \ y = Cx + Du$

change coordinates in $\mathbb{R}^n$ to $\tilde{x}$, with $x = T\tilde{x}$

then

$$\dot{x} = T^{-1}\dot{x} = T^{-1}(Ax + Bu) = T^{-1}AT\tilde{x} + T^{-1}Bu$$

hence LDS can be expressed as

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \quad y = \tilde{C}\tilde{x} + \tilde{D}u$$

where

$$\tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B, \quad \tilde{C} = CT, \quad \tilde{D} = D$$

TF is same (since $u, y$ aren’t affected):

$$\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} = C(sI - A)^{-1}B + D$$
Standard forms for LDS

can change coordinates to put $A$ in various forms (diagonal, real modal, Jordan …)

e.g., to put LDS in diagonal form, find $T$ s.t.

$$T^{-1}AT = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

write

$$T^{-1}B = \begin{bmatrix} \tilde{b}_1^T \\ \vdots \\ \tilde{b}_n^T \end{bmatrix}, \quad CT = [\tilde{c}_1 \ \cdots \ \tilde{c}_n]$$