Linear dynamical systems with inputs & outputs

- inputs & outputs: interpretations
- transfer function
- impulse and step responses
- examples

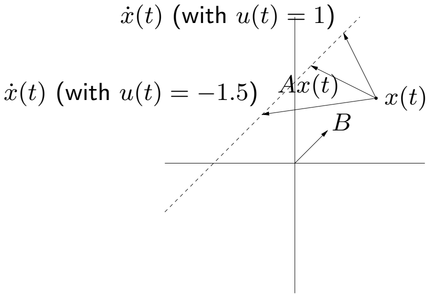
Inputs & outputs

recall continuous-time time-invariant LDS has form

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

- Ax is called the *drift term* (of $\dot{x}$)
- Bu is called the input term (of $\dot{x}$)

picture, with $B \in \mathbb{R}^{2\times 1}$:



Interpretations

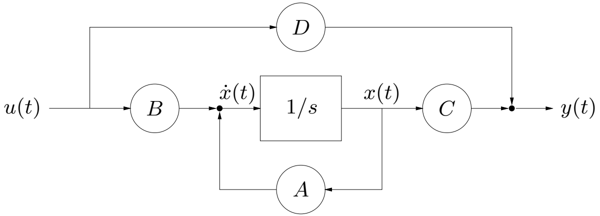
write $\dot{x} = Ax + b_1 u_1 + \cdots + b_m u_m$, where $B = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}$

- state derivative is sum of autonomous term (Ax) and one term per input ($b_i u_i$)
- each input u_i gives another degree of freedom for $\dot{x}$ (assuming columns of B independent)

write $\dot{x} = Ax + Bu$ as $\dot{x}_i = \tilde{a}_i^\mathsf{T} x + \tilde{b}_i^\mathsf{T} u$, where $\tilde{a}_i^\mathsf{T}$, $\tilde{b}_i^\mathsf{T}$ are the rows of A, B

- ith state derivative is linear function of state x and input u

Block diagram

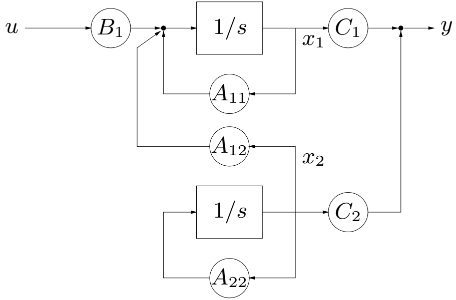


- A_{ij} is gain factor from state x_j into integrator i
- B_{ij} is gain factor from input u_j into integrator i
- C_{ij} is gain factor from state x_j into output y_i
- D_{ij} is gain factor from input u_j into output y_i

Structure

interesting when there is structure, *e.g.*, with $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



- x_2 is not affected by input u, *i.e.*, x_2 propagates autonomously
- x_2 affects y directly and through x_1

Transfer function

take Laplace transform of $\dot{x} = Ax + Bu$:

$$sX(s) - x(0) = AX(s) + BU(s)$$

hence

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

so

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau)\,d\tau$$

- $e^{tA}x(0)$ is the unforced or autonomous response
- $e^{tA}B$ is called the input-to-state impulse response or impulse matrix
- $(sI - A)^{-1}B$ is called the *input-to-state transfer function* or *transfer matrix*

Transfer function

with $y = Cx + Du$ we have:

$$Y(s) = C(sI - A)^{-1}x(0) + (C(sI - A)^{-1}B + D)U(s)$$

so

$$y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau)\,d\tau + Du(t)$$

- output term $Ce^{tA}x(0)$ due to initial condition
- $H(s) = C(sI - A)^{-1}B + D$ is called the *transfer function* or *transfer matrix*
- $h(t) = Ce^{tA}B + D\delta(t)$ is called the *impulse response* or *impulse matrix* (δ is the Dirac delta function)

with zero initial condition we have:

$$Y(s) = H(s)U(s), \qquad y = h * u$$

where $*$ is convolution (of matrix valued functions)

intepretation:

- H_{ij} is transfer function from input u_j to output y_i

Impulse response

impulse response $h(t) = Ce^{tA}B + D\delta(t)$

with $x(0) = 0$, $y = h * u$, *i.e.*,

$$y_i(t) = \sum_{j=1}^{m} \int_0^t h_{ij}(t-\tau)u_j(\tau)\, d\tau$$

interpretations:

- $h_{ij}(t)$ is impulse response from jth input to ith output
- $h_{ij}(t)$ gives $y_i(t)$ when $u(t) = e_j\delta(t)$
- $h_{ij}(\tau)$ shows how dependent output i is, on what input j was, τ seconds ago
- i indexes output; j indexes input; τ indexes time lag

Step response

the *step response* or *step matrix* is given by

$$s(t) = \int_0^t h(\tau)\, d\tau$$

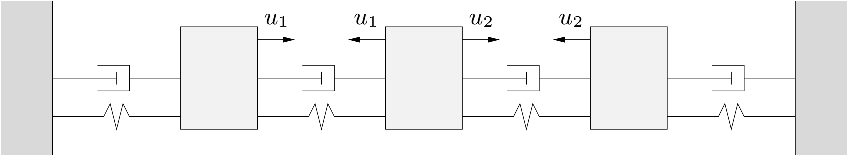
interpretations:

- $s_{ij}(t)$ is step response from jth input to ith output
- $s_{ij}(t)$ gives y_i when $u = e_j$ for $t \geq 0$

for invertible A, we have

$$s(t) = CA^{-1}\left(e^{tA} - I\right)B + D$$

Example 1



- unit masses, springs, dampers
- u_1 is tension between 1st & 2nd masses
- u_2 is tension between 2nd & 3rd masses
- $y \in \mathbb{R}^3$ is displacement of masses 1,2,3
- $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

Example 1

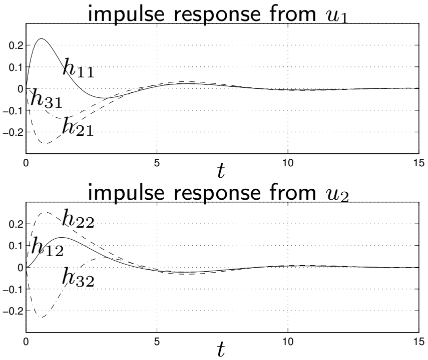
system is:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 & 1 & 0 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

eigenvalues of A are

$$-1.71 \pm i0.71, \qquad -1.00 \pm i1.00, \qquad -0.29 \pm i0.71$$

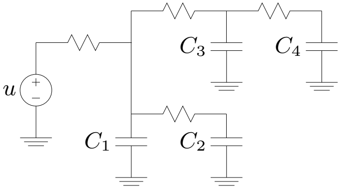
Example 1: Impulse response



roughly speaking:

- impulse at u_1 affects third mass less than other two
- impulse at u_2 affects first mass later than other two

Example 2: Circuit



- $u(t) \in \mathbb{R}$ is input (drive) voltage
- x_i is voltage across C_i
- output is state: $y = x$
- unit resistors, unit capacitors
- step response matrix shows delay to each node

Example 2: Circuit

system is

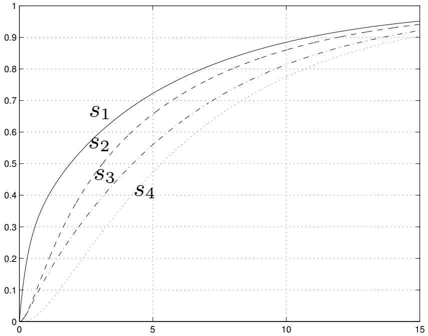
$$\dot{x} = \begin{bmatrix} -3 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \qquad y = x$$

eigenvalues of A are

$$-0.17, \qquad -0.66, \qquad -2.21, \qquad -3.96$$

Example 2: Circuit

step response matrix $s(t) \in \mathbb{R}^{4\times 1}$:



- shortest delay to x_1; longest delay to x_4
- delays ≈ 10, consistent with slowest ($i.e.$, dominant) eigenvalue -0.17

DC or static gain matrix

- transfer function at $s = 0$ is $H(0) = -CA^{-1}B + D \in \mathbb{R}^{m \times p}$
- DC transfer function describes system under *static* conditions, $i.e.$, x, u, y constant:

  $$0 = \dot{x} = Ax + Bu, \qquad y = Cx + Du$$

  eliminate x to get $y = H(0)u$

- if system is stable,

  $$H(0) = \int_0^\infty h(t)\, dt = \lim_{t \to \infty} s(t)$$

  (recall: $H(s) = \int_0^\infty e^{-st} h(t)\, dt$, $s(t) = \int_0^t h(\tau)\, d\tau$)

  if $u(t) \to u_\infty \in \mathbb{R}^m$, then $y(t) \to y_\infty \in \mathbb{R}^p$ where $y_\infty = H(0)u_\infty$

DC gain matrix

DC gain matrix for example 1 (springs):

$$H(0) = \begin{bmatrix} 1/4 & 1/4 \\ -1/2 & 1/2 \\ -1/4 & -1/4 \end{bmatrix}$$

DC gain matrix for example 2 (RC circuit):

$$H(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(do these make sense?)

Discretization with piecewise constant inputs

linear system $\dot{x} = Ax + Bu$, $y = Cx + Du$

suppose $u_d : \mathbb{Z}_+ \to \mathbb{R}^m$ is a sequence, and

$$u(t) = u_d(k) \qquad \text{for} \quad kh \le t < (k+1)h, \quad k = 0, 1, \ldots$$

define sequences

$$x_d(k) = x(kh), \qquad y_d(k) = y(kh), \qquad k = 0, 1, \ldots$$

- $h > 0$ is called the *sample interval* (for x and y) or *update interval* (for u)
- u is piecewise constant (called *zero-order-hold*)
- x_d, y_d are sampled versions of x, y

Discretization with piecewise constant inputs

$$\begin{aligned}
x_d(k+1) &= x((k+1)h) \\
&= e^{hA}x(kh) + \int_0^h e^{\tau A} Bu((k+1)h - \tau)\, d\tau \\
&= e^{hA}x_d(k) + \left(\int_0^h e^{\tau A}\, d\tau\right) B\, u_d(k)
\end{aligned}$$

x_d, u_d, and y_d satisfy discrete-time LDS equations

$$x_d(k+1) = A_d x_d(k) + B_d u_d(k), \qquad y_d(k) = C_d x_d(k) + D_d u_d(k)$$

where

$$A_d = e^{hA}, \qquad B_d = \left(\int_0^h e^{\tau A}\, d\tau\right) B, \qquad C_d = C, \qquad D_d = D$$

called *discretized system*. If A is invertible, we can express integral as

$$\int_0^h e^{\tau A}\, d\tau = A^{-1}\left(e^{hA} - I\right)$$

Stability of discretization

stability: if eigenvalues of A are $\lambda_1, \ldots, \lambda_n$, then eigenvalues of A_d are $e^{h\lambda_1}, \ldots, e^{h\lambda_n}$

discretization preserves stability properties since

$$\Re\lambda_i < 0 \quad \Leftrightarrow \quad |e^{h\lambda_i}| < 1$$

for $h > 0$

Extensions and variations

- *offsets:* updates for u and sampling of x, y are offset in time
- *multirate:* u_i updated, y_i sampled at different intervals
  (usually integer multiples of a common interval h)

both very common in practice

Causality

interpretation of

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau)\,d\tau$$

$$y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau)\,d\tau + Du(t)$$

for $t \geq 0$:

current state ($x(t)$) and output ($y(t)$) depend on *past* input ($u(\tau)$ for $\tau \leq t$)

$i.e.$, mapping from input to state and output is *causal* (with fixed *initial* state)

now consider fixed *final* state $x(T)$: for $t \leq T$,

$$x(t) = e^{(t-T)A}x(T) + \int_T^t e^{(t-\tau)A}Bu(\tau)\,d\tau,$$

$i.e.$, current state (and output) depend on future input!

so for fixed final condition, same system is anti-causal

Idea of state

$x(t)$ is called *state* of system at time t since:

- future output depends only on current state and future input
- future output depends on past input only through current state
- state summarizes effect of past inputs on future output
- state is bridge between past inputs and future outputs

Change of coordinates

start with LDS $\dot{x} = Ax + Bu$, $y = Cx + Du$

change coordinates in $\mathbb{R}^n$ to $\tilde{x}$, with $x = T\tilde{x}$

then

$$\dot{\tilde{x}} = T^{-1}\dot{x} = T^{-1}(Ax + Bu) = T^{-1}AT\tilde{x} + T^{-1}Bu$$

hence LDS can be expressed as

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \qquad y = \tilde{C}\tilde{x} + \tilde{D}u$$

where

$$\tilde{A} = T^{-1}AT, \qquad \tilde{B} = T^{-1}B, \qquad \tilde{C} = CT, \qquad \tilde{D} = D$$

TF is same (since u, y aren't affected):

$$\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} = C(sI - A)^{-1}B + D$$

Standard forms for LDS

can change coordinates to put A in various forms (diagonal, real modal, Jordan $\ldots$)

$e.g.$, to put LDS in *diagonal form*, find T s.t.

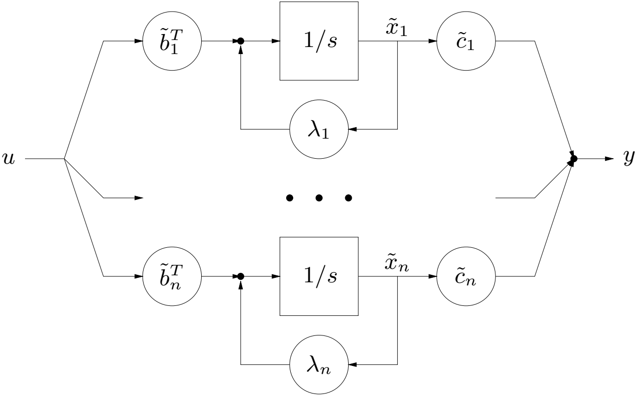
$$T^{-1}AT = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$$

write

$$T^{-1}B = \begin{bmatrix} \tilde{b}_1^\mathsf{T} \\ \vdots \\ \tilde{b}_n^\mathsf{T} \end{bmatrix}, \qquad CT = \begin{bmatrix} \tilde{c}_1 & \cdots & \tilde{c}_n \end{bmatrix}$$

Diagonal form

$$\dot{\tilde{x}}_i = \lambda_i \tilde{x}_i + \tilde{b}_i^\mathsf{T} u, \qquad y = \sum_{i=1}^{n} \tilde{c}_i \tilde{x}_i$$

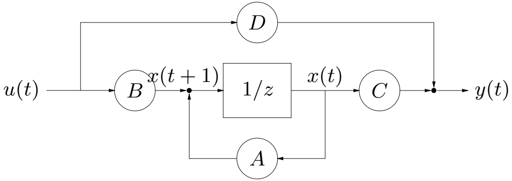


(here we assume $D = 0$)

Discrete-time systems

discrete-time LDS:

$$x(t+1) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$



- only difference w/cts-time: z instead of s
- interpretation of z^{-1} block:
  - unit delayor (shifts sequence back in time one epoch)
  - latch (plus small delay to avoid race condition)

Explicit solution

we have:

$$x(1) = Ax(0) + Bu(0),$$

$$\begin{aligned} x(2) &= Ax(1) + Bu(1) \\ &= A^2x(0) + ABu(0) + Bu(1), \end{aligned}$$

and in general, for $t \in \mathbb{Z}_+$,

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)} Bu(\tau)$$

hence

$$y(t) = CA^t x(0) + h * u$$

where $*$ is discrete-time convolution and

$$h(t) = \begin{cases} D, & t = 0 \\ CA^{t-1}B, & t > 0 \end{cases}$$

is the impulse response

$\mathcal{Z}$-transform

suppose $w \in \mathbb{R}^{p\times q}$ is a sequence (discrete-time signal), $i.e.$,

$$w : \mathbb{Z}_+ \to \mathbb{R}^{p\times q}$$

recall $\mathcal{Z}$-transform $W = \mathcal{Z}(w)$:

$$W(z) = \sum_{t=0}^{\infty} z^{-t} w(t)$$

where $W : D \subseteq \mathbb{C} \to \mathbb{C}^{p\times q}$ (D is domain of W)

time-advanced or shifted signal v:

$$v(t) = w(t+1), \qquad t = 0, 1, \ldots$$

$\mathcal{Z}$-transform of time-advanced signal:

$$\begin{aligned} V(z) &= \sum_{t=0}^{\infty} z^{-t} w(t+1) \\ &= z \sum_{t=1}^{\infty} z^{-t} w(t) \\ &= zW(z) - zw(0) \end{aligned}$$

Discrete-time transfer function

take $\mathcal{Z}$-transform of system equations

$$x(t+1) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$

yields

$$zX(z) - zx(0) = AX(z) + BU(z), \qquad Y(z) = CX(z) + DU(z)$$

solve for $X(z)$ to get

$$X(z) = (zI - A)^{-1} zx(0) + (zI - A)^{-1} BU(z)$$

(note extra z in first term!)

hence

$$Y(z) = H(z)U(z) + C(zI - A)^{-1} zx(0)$$

where $H(z) = C(zI - A)^{-1}B + D$ is the *discrete-time transfer function*. Note power series expansion of resolvent:

$$(zI - A)^{-1} = z^{-1}I + z^{-2}A + z^{-3}A^2 + \cdots$$