Linear dynamical systems with inputs & outputs

- inputs & outputs: interpretations
- transfer function
- impulse and step responses
- examples
### Inputs & outputs

recall continuous-time time-invariant LDS has form

\[
\dot{x} = Ax + Bu, \quad y = Cx + Du
\]

- \(Ax\) is called the drift term (of \(\dot{x}\))
- \(Bu\) is called the input term (of \(\dot{x}\))

picture, with \(B \in \mathbb{R}^{2 \times 1}\):

- \(\dot{x}(t) \text{ (with } u(t) = 1)\)
- \(\dot{x}(t) \text{ (with } u(t) = -1.5)\)
write $\dot{x} = Ax + b_1 u_1 + \cdots + b_m u_m$, where $B = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}$

- state derivative is sum of autonomous term ($Ax$) and one term per input ($b_i u_i$)

- each input $u_i$ gives another degree of freedom for $\dot{x}$ (assuming columns of $B$ independent)

write $\dot{x} = Ax + Bu$ as $\dot{x}_i = \tilde{a}_i^T x + \tilde{b}_i^T u$, where $\tilde{a}_i^T, \tilde{b}_i^T$ are the rows of $A, B$

- $i$th state derivative is linear function of state $x$ and input $u$
\[ u(t) \rightarrow B \rightarrow \dot{x}(t) \rightarrow 1/s \rightarrow x(t) \rightarrow C \rightarrow y(t)\]

- \( A_{ij} \) is gain factor from state \( x_j \) into integrator \( i \)
- \( B_{ij} \) is gain factor from input \( u_j \) into integrator \( i \)
- \( C_{ij} \) is gain factor from state \( x_j \) into output \( y_i \)
- \( D_{ij} \) is gain factor from input \( u_j \) into output \( y_i \)
Structure

Interesting when there is structure, e.g., with $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- $x_2$ is not affected by input $u$, i.e., $x_2$ propagates autonomously

- $x_2$ affects $y$ directly and through $x_1$
Transfer function

take Laplace transform of $\dot{x} = Ax + Bu$:

$$sX(s) - x(0) = AX(s) + BU(s)$$

hence

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

so

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau) \, d\tau$$

- $e^{tA}x(0)$ is the unforced or autonomous response
- $e^{tA}B$ is called the input-to-state impulse response or impulse matrix
- $(sI - A)^{-1}B$ is called the input-to-state transfer function or transfer matrix
**Transfer function**

with $y = Cx + Du$ we have:

$$Y(s) = C(sI - A)^{-1}x(0) + (C(sI - A)^{-1}B + D)U(s)$$

so

$$y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau) \, d\tau + Du(t)$$

- output term $Ce^{tA}x(0)$ due to initial condition
- $H(s) = C(sI - A)^{-1}B + D$ is called the transfer function or transfer matrix
- $h(t) = Ce^{tA}B + D\delta(t)$ is called the impulse response or impulse matrix ($\delta$ is the Dirac delta function)

with zero initial condition we have:

$$Y(s) = H(s)U(s), \quad y = h \ast u$$

where $\ast$ is convolution (of matrix valued functions)

interpretation:

- $H_{ij}$ is transfer function from input $u_j$ to output $y_i$
Impulse response

impulse response $h(t) = C e^{tA} B + D\delta(t)$

with $x(0) = 0$, $y = h * u$, i.e.,

$$y_i(t) = \sum_{j=1}^{m} \int_{0}^{t} h_{ij}(t - \tau) u_j(\tau) \, d\tau$$

interpretations:

$\blacktriangleright$ $h_{ij}(t)$ is impulse response from $j$th input to $i$th output

$\blacktriangleright$ $h_{ij}(t)$ gives $y_i(t)$ when $u(t) = e_j \delta(t)$

$\blacktriangleright$ $h_{ij}(\tau)$ shows how dependent output $i$ is, on what input $j$ was, $\tau$ seconds ago

$\blacktriangleright$ $i$ indexes output; $j$ indexes input; $\tau$ indexes time lag
Step response

the *step response* or *step matrix* is given by

\[ s(t) = \int_0^t h(\tau) \, d\tau \]

interpretations:

- \( s_{ij}(t) \) is step response from \( j \)th input to \( i \)th output
- \( s_{ij}(t) \) gives \( y_i \) when \( u = e_j \) for \( t \geq 0 \)

for invertible \( A \), we have

\[ s(t) = CA^{-1} \left( e^{tA} - I \right) B + D \]
Example 1

- unit masses, springs, dampers
- $u_1$ is tension between 1st & 2nd masses
- $u_2$ is tension between 2nd & 3rd masses
- $y \in \mathbb{R}^3$ is displacement of masses 1,2,3
- $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$
Example 1

system is:

\[
\dot{x} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-2 & 1 & 0 & -2 & 1 & 0 \\
1 & -2 & 1 & 1 & -2 & 1 \\
0 & 1 & -2 & 0 & 1 & -2 \\
\end{bmatrix} x + \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
-1 & 1 \\
0 & -1 \\
\end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\]

eigenvalues of \( A \) are

\[-1.71 \pm i0.71, \quad -1.00 \pm i1.00, \quad -0.29 \pm i0.71\]
Example 1: Impulse response

- impulse response from $u_1$
- impulse response from $u_2$

roughly speaking:

- impulse at $u_1$ affects third mass less than other two
- impulse at $u_2$ affects first mass later than other two
Example 2: Circuit

- $u(t) \in \mathbb{R}$ is input (drive) voltage
- $x_i$ is voltage across $C_i$
- output is state: $y = x$
- unit resistors, unit capacitors
- step response matrix shows delay to each node
Example 2: Circuit

system is

\[
\dot{x} = \begin{bmatrix}
-3 & 1 & 1 & 0 \\
1 & -1 & 0 & 0 \\
1 & 0 & -2 & 1 \\
0 & 0 & 1 & -1 \\
\end{bmatrix} x + \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
\end{bmatrix} u, \quad y = x
\]

eigenvalues of \( A \) are

\[-0.17, \quad -0.66, \quad -2.21, \quad -3.96\]
Example 2: Circuit

Step response matrix \( s(t) \in \mathbb{R}^{4 \times 1} \):

- Shortest delay to \( x_1 \); longest delay to \( x_4 \)
- Delays \( \approx 10 \), consistent with slowest (i.e., dominant) eigenvalue \(-0.17\)
DC or static gain matrix

- transfer function at \( s = 0 \) is \( H(0) = -CA^{-1}B + D \in \mathbb{R}^{m \times p} \)

- DC transfer function describes system under static conditions, i.e., \( x, u, y \) constant:

\[
0 = \dot{x} = Ax + Bu, \quad y = Cx + Du
\]

eliminate \( x \) to get \( y = H(0)u \)

- if system is stable, \( H(0) = \int_0^\infty h(t) \, dt = \lim_{t \to \infty} s(t) \)

(recall: \( H(s) = \int_0^\infty e^{-st}h(t) \, dt \), \( s(t) = \int_0^t h(\tau) \, d\tau \))

if \( u(t) \to u_\infty \in \mathbb{R}^m \), then \( y(t) \to y_\infty \in \mathbb{R}^p \) where \( y_\infty = H(0)u_\infty \)
DC gain matrix

DC gain matrix for example 1 (springs):

\[ H(0) = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \]

DC gain matrix for example 2 (RC circuit):

\[ H(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]

(do these make sense?)
Discretization with piecewise constant inputs

linear system $\dot{x} = Ax + Bu$, $y = Cx + Du$

suppose $u_d : \mathbb{Z}_+ \rightarrow \mathbb{R}^m$ is a sequence, and

$$u(t) = u_d(k) \quad \text{for} \quad kh \leq t < (k + 1)h, \; k = 0, 1, \ldots$$

define sequences

$$x_d(k) = x(kh), \quad y_d(k) = y(kh), \quad k = 0, 1, \ldots$$

- $h > 0$ is called the sample interval (for $x$ and $y$) or update interval (for $u$)
- $u$ is piecewise constant (called zero-order-hold)
- $x_d, y_d$ are sampled versions of $x, y$
Discretization with piecewise constant inputs

\[ x_d(k + 1) = x((k + 1)h) \]
\[ = e^{hA} x(kh) + \int_0^h e^{\tau A} Bu((k + 1)h - \tau) \, d\tau \]
\[ = e^{hA} x_d(k) + \left( \int_0^h e^{\tau A} \, d\tau \right) B \ u_d(k) \]

\( x_d, \ u_d, \) and \( y_d \) satisfy discrete-time LDS equations

\[ x_d(k + 1) = A_d x_d(k) + B_d u_d(k), \quad y_d(k) = C_d x_d(k) + D_d u_d(k) \]

where

\[ A_d = e^{hA}, \quad B_d = \left( \int_0^h e^{\tau A} \, d\tau \right) B, \quad C_d = C, \quad D_d = D \]

called \textit{discretized system}. If \( A \) is invertible, we can express integral as

\[ \int_0^h e^{\tau A} \, d\tau = A^{-1} (e^{hA} - I) \]
Stability of discretization

**Stability:** if eigenvalues of $A$ are $\lambda_1, \ldots, \lambda_n$, then eigenvalues of $A_d$ are $e^{h\lambda_1}, \ldots, e^{h\lambda_n}$.

Discretization preserves stability properties since

$$\Re \lambda_i < 0 \iff |e^{h\lambda_i}| < 1$$

for $h > 0$.
Extensions and variations

- **offsets**: updates for $u$ and sampling of $x$, $y$ are offset in time

- **multirate**: $u_i$ updated, $y_i$ sampled at different intervals
  (usually integer multiples of a common interval $h$)

both very common in practice
Causality

interpretation of

\[ x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau) \, d\tau \]

\[ y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau) \, d\tau + Du(t) \]

for \( t \geq 0 \):

*current* state \( (x(t)) \) and output \( (y(t)) \) depend on *past* input \( (u(\tau) \text{ for } \tau \leq t) \)

i.e., mapping from input to state and output is *causal* (with fixed *initial* state)

now consider fixed *final* state \( x(T) \): for \( t \leq T \),

\[ x(t) = e^{(t-T)A}x(T) + \int_T^t e^{(t-\tau)A}Bu(\tau) \, d\tau, \]

i.e., current state (and output) depend on future input!

so for fixed final condition, same system is anti-causal
Idea of state

$x(t)$ is called *state* of system at time $t$ since:

- future output depends only on current state and future input
- future output depends on past input only through current state
- state summarizes effect of past inputs on future output
- state is bridge between past inputs and future outputs
Change of coordinates

start with LDS $\dot{x} = Ax + Bu$, $y = Cx + Du$

change coordinates in $\mathbb{R}^n$ to $\tilde{x}$, with $x = T\tilde{x}$

then

$\dot{x} = T^{-1} \dot{\tilde{x}} = T^{-1} (Ax + Bu) = T^{-1} AT\tilde{x} + T^{-1} Bu$

hence LDS can be expressed as

$\dot{x} = \tilde{A}\tilde{x} + \tilde{B}u$, $y = \tilde{C}\tilde{x} + \tilde{D}u$

where

$\tilde{A} = T^{-1} AT$, $\tilde{B} = T^{-1} B$, $\tilde{C} = CT$, $\tilde{D} = D$

TF is same (since $u$, $y$ aren’t affected):

$\tilde{C}(sI - \tilde{A})^{-1} \tilde{B} + \tilde{D} = C(sI - A)^{-1} B + D$
Standard forms for LDS

can change coordinates to put $A$ in various forms (diagonal, real modal, Jordan ...)

* e.g., to put LDS in *diagonal form*, find $T$ s.t.

$$T^{-1}AT = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

write

$$T^{-1}B = \begin{bmatrix}
\tilde{b}_1^T \\
\vdots \\
\tilde{b}_n^T
\end{bmatrix}, \quad
CT = \begin{bmatrix}
\tilde{c}_1 \\
\cdots \\
\tilde{c}_n
\end{bmatrix}$$
Diagonal form

\[
\dot{x}_i = \lambda_i \tilde{x}_i + \tilde{b}_i^T u, \quad y = \sum_{i=1}^{n} \tilde{c}_i \tilde{x}_i
\]

(here we assume \( D = 0 \))
Discrete-time systems

discrete-time LDS:

\[ x(t + 1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) \]

- only difference w/cts-time: \( z \) instead of \( s \)
- interpretation of \( z^{-1} \) block:
  - unit delayor (shifts sequence back in time one epoch)
  - latch (plus small delay to avoid race condition)
Explicit solution
we have:

\[ x(1) = Ax(0) + Bu(0), \]

\[ x(2) = A x(1) + Bu(1) = A^2 x(0) + ABu(0) + Bu(1), \]

and in general, for \( t \in \mathbb{Z}_+ \),

\[ x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{t-1-\tau} Bu(\tau) \]

hence

\[ y(t) = CA^t x(0) + h * u \]

where \( * \) is discrete-time convolution and

\[ h(t) = \begin{cases} 
D, & t = 0 \\
CA^{t-1} B, & t > 0 
\end{cases} \]

is the impulse response
Block Toeplitz matrices

we have

\[
\begin{bmatrix}
y(0) \\
y(1) \\
y(2) \\
\vdots \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
D & & & & \\
CB & D & & & \\
CAB & CB & D & & \\
& & & & \\
CA^{t-1}B & CA^{t-2}B & \cdots & CB & D
\end{bmatrix}
\begin{bmatrix}
u(0) \\
u(1) \\
u(2) \\
\vdots \\
u(t)
\end{bmatrix} +
\begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^t
\end{bmatrix} x(0)
\]

► this matrix gives the output sequence \(y(0), y(1), \ldots\) in terms of the input sequence \(u(0), u(1), \ldots\) and the initial state \(x(0)\)

► block Toeplitz means blocks are constant along diagonals from top-left to bottom right

► we can use this to find controllers and estimators