Linear functions
Linear equations

consider system of linear equations

\[ y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \]
\[ y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \]
\[ \vdots \]
\[ y_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \]

can be written in matrix form as \( y = Ax \), where

\[
y = \begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_m
\end{bmatrix} \quad A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} \quad x = \begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}
\]
Linear functions

A function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is linear if

1. \( f(x + y) = f(x) + f(y), \forall x, y \in \mathbb{R}^n \)
2. \( f(\alpha x) = \alpha f(x), \forall x \in \mathbb{R}^n \forall \alpha \in \mathbb{R} \)

I.e., superposition holds
Matrix multiplication function

- consider function \( f : \mathbb{R}^n \to \mathbb{R}^m \) given by \( f(x) = Ax \), where \( A \in \mathbb{R}^{m \times n} \)

- matrix multiplication function \( f \) is linear

- **converse** is true: any linear function \( f : \mathbb{R}^n \to \mathbb{R}^m \) can be written as \( f(x) = Ax \) for some \( A \in \mathbb{R}^{m \times n} \)

- representation via matrix multiplication is unique: for any linear function \( f \) there is only one matrix \( A \) for which \( f(x) = Ax \) for all \( x \)

- \( y = Ax \) is a concrete representation of a generic linear function
Interpretations of $y = Ax$

- $y$ is measurement or observation; $x$ is unknown to be determined
- $x$ is ‘input’ or ‘action’; $y$ is ‘output’ or ‘result’
- $y = Ax$ defines a function or transformation that maps $x \in \mathbb{R}^n$ into $y \in \mathbb{R}^m$
Interpretation of $a_{ij}$

$$y_i = \sum_{j=1}^{n} a_{ij} x_j$$

$a_{ij}$ is gain factor from $j$th input ($x_j$) to $i$th output ($y_i$)

- $i$th row of $A$ concerns $i$th output
- $j$th column of $A$ concerns $j$th input
- $a_{27} = 0$ means 2nd output ($y_2$) doesn’t depend on 7th input ($x_7$)
- $|a_{31}| \gg |a_{3j}|$ for $j \neq 1$ means $y_3$ depends mainly on $x_1$
- $|a_{52}| \gg |a_{i2}|$ for $i \neq 5$ means $x_2$ affects mainly $y_5$
- $A$ is lower triangular, *i.e.*, $a_{ij} = 0$ for $i < j$, means $y_i$ only depends on $x_1, \ldots, x_i$
- $A$ is diagonal, *i.e.*, $a_{ij} = 0$ for $i \neq j$, means $i$th output depends only on $i$th input

more generally, sparsity pattern of $A$, *i.e.*, list of zero/nonzero entries of $A$, shows which $x_j$ affect which $y_i$