Interpreting Linear Equations

Broad categories of applications

linear model or function $y = Ax$

some broad categories of applications:

- estimation or inversion
- control or design
- mapping or transformation

(this list is not exclusive; can have combinations . . .)

Estimation or inversion

$$y = Ax$$

- y_i is ith measurement or sensor reading (which we know)
- x_j is jth parameter to be estimated or determined
- a_{ij} is sensitivity of ith sensor to jth parameter

sample problems:

- find x, given y
- find all x's that result in y (*i.e.*, all x's consistent with measurements)
- if there is no x such that $y = Ax$, find x s.t. $y \approx Ax$ (*i.e.*, if the sensor readings are inconsistent, find x which is almost consistent)

Control or design

$$y = Ax$$

- x is vector of design parameters or inputs (which we can choose)
- y is vector of results, or outcomes
- A describes how input choices affect results

sample problems:

- find x so that $y = y_{\text{des}}$
- find all x's that result in $y = y_{\text{des}}$ (*i.e.*, find all designs that meet specifications)
- among x's that satisfy $y = y_{\text{des}}$, find a small one (*i.e.*, find a small or efficient x that meets specifications)

Mapping or transformation

- x is mapped or transformed to y by linear function $y = Ax$

sample problems:

- determine if there is an x that maps to a given y
- (if possible) find *an* x that maps to y
- find *all* x's that map to a given y
- if there is only one x that maps to y, find it ($i.e.$, decode or undo the mapping)

Matrix multiplication as mixture of columns

write $A \in \mathbb{R}^{m \times n}$ in terms of its columns

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

where $a_j \in \mathbb{R}^m$. Then then $y = Ax$ means

$$y = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

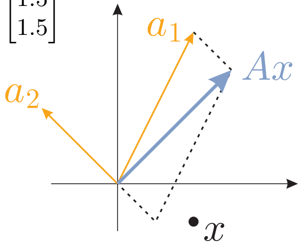
(x_j's are scalars, a_j's are m-vectors)

- y is a (linear) combination or mixture of the columns of A
- coefficients of x give coefficients of mixture
- each column of A represents an *actuator*

Geometric interpretation of control

example: $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, $\quad x = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$, $\quad y = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$

$$Ax = a_1 + (-0.5)a_2 = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$$



another example:

$$a_j = Ae_j$$

where e_j is the $j$the unit vector:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \qquad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ n \end{bmatrix}$$

- jth column of A gives response to unit jth input

Matrix multiplication as inner product with rows

write A in terms of its rows:

$$A = \begin{bmatrix} \tilde{a}_1^\mathsf{T} \\ \tilde{a}_2^\mathsf{T} \\ \vdots \\ \tilde{a}_m^\mathsf{T} \end{bmatrix}$$

where $\tilde{a}_i \in \mathbb{R}^n$

then $y = Ax$ can be written as

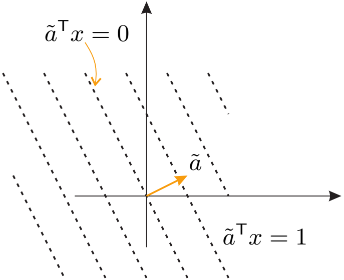
$$y = \begin{bmatrix} \tilde{a}_1^\mathsf{T} x \\ \tilde{a}_2^\mathsf{T} x \\ \vdots \\ \tilde{a}_m^\mathsf{T} x \end{bmatrix}$$

- $y_i = \tilde{a}_i^\mathsf{T} x$, so that y_i is inner product of ith row of A with x
- each row of A represents a *sensor*

Geometric interpretation of estimation

$$b_i^T x = \text{constant}$$

is a (hyper-)plane in $\mathbb{R}^n$ normal to b_i.

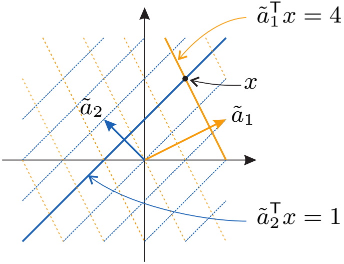


if $Ax = y$ then x is on intersection of hyperplanes $b_i^T x = y_i$

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$y = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

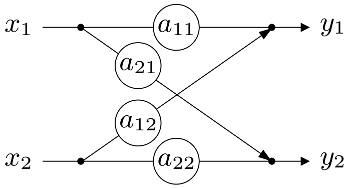


Block diagram representation

$y = Ax$ can be represented by a *signal flow graph* or *block diagram* $e.g.$ for $m = n = 2$, we represent

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

as



- a_{ij} is the gain along the path from jth input to ith output
- (by not drawing paths with zero gain) shows sparsity structure of A ($e.g.$, diagonal, block upper triangular, arrow ...)

Example: block upper triangular matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

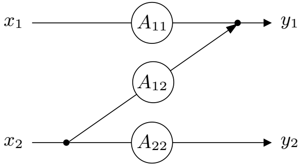
where $A_{11} \in \mathbb{R}^{m_1 \times n_1}$, $A_{12} \in \mathbb{R}^{m_1 \times n_2}$, $A_{21} \in \mathbb{R}^{m_2 \times n_1}$, $A_{22} \in \mathbb{R}^{m_2 \times n_2}$

partition x and y conformably, (so that $x_1 \in \mathbb{R}^{n_1}$,$x_2 \in \mathbb{R}^{n_2}$ $y_1 \in \mathbb{R}^{m_1}$, $y_2 \in \mathbb{R}^{m_2}$)

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

then

$$\begin{aligned} y_1 &= A_{11}x_1 + A_{12}x_2 \\ y_2 &= A_{22}x_2, \end{aligned}$$



...no path from x_1 to y_2, so y_2 doesn't depend on x_1

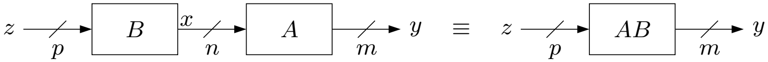
Matrix multiplication as composition

for $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, $C = AB \in \mathbb{R}^{m \times p}$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

composition interpretation

$y = Cz$ represents composition of $y = Ax$ and $x = Bz$



(note that B is on left in block diagram)

Column and row interpretations

can write product $C = AB$ as

$$C = \begin{bmatrix} c_1 & \cdots & c_p \end{bmatrix} = AB = \begin{bmatrix} Ab_1 & \cdots & Ab_p \end{bmatrix}$$

$i.e.$, ith column of C is A acting on ith column of B

similarly we can write

$$C = \begin{bmatrix} \tilde{c}_1^\mathsf{T} \\ \vdots \\ \tilde{c}_m^\mathsf{T} \end{bmatrix} = AB = \begin{bmatrix} \tilde{a}_1^\mathsf{T} B \\ \vdots \\ \tilde{a}_m^\mathsf{T} B \end{bmatrix}$$

$i.e.$, ith row of C is ith row of A acting (on left) on B

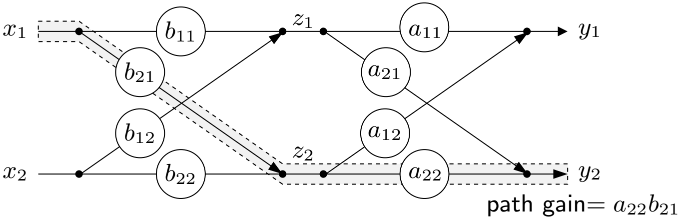
Inner product interpretation

$$c_{ij} = \tilde{a}_i^\mathsf{T} b_j = \langle \tilde{a}_i, b_j \rangle$$

$i.e.$, entries of C are inner products of rows of A and columns of B

- $c_{ij} = 0$ means ith row of A is orthogonal to jth column of B
- **Gram matrix** of vectors $f_1, \ldots, f_n$ defined as $G_{ij} = f_i^\mathsf{T} f_j$
  (gives inner product of each vector with the others)
- $G = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}^\mathsf{T} \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}$

Matrix multiplication interpretation via paths



- $a_{ik}b_{kj}$ is gain of path from input j to output i via k
- c_{ij} is sum of gains over *all* paths from input j to output i