Gauss-Newton method

- nonlinear least-squares
- Gauss-Newton method
Nonlinear least-squares

**nonlinear least-squares (NLLS) problem:** find $x \in \mathbb{R}^n$ that minimizes

$$\|r(x)\|^2 = \sum_{i=1}^{m} r_i(x)^2,$$

where $r : \mathbb{R}^n \rightarrow \mathbb{R}^m$

- $r(x)$ is a vector of ‘residuals’
- reduces to (linear) least-squares if $r(x) = Ax - y$
Position estimation from ranges

estimate position $x \in \mathbb{R}^2$ from approximate distances to beacons at locations $b_1, \ldots, b_m \in \mathbb{R}^2$ \textit{without} linearizing

- we measure $\rho_i = \|x - b_i\| + v_i$
  ($v_i$ is range error, unknown but assumed small)

- NLLS estimate: choose $\hat{x}$ to minimize

$$
\sum_{i=1}^{m} r_i(x)^2 = \sum_{i=1}^{m} (\rho_i - \|x - b_i\|)^2
$$
Gauss-Newton method for NLLS

\textbf{NLLS:} find $x \in \mathbb{R}^n$ that minimizes $\|r(x)\|^2 = \sum_{i=1}^{m} r_i(x)^2$, where $r : \mathbb{R}^n \rightarrow \mathbb{R}^m$

- in general, very hard to solve exactly
- many good heuristics to compute \textit{locally optimal} solution

\textbf{Gauss-Newton method:}

given starting guess for $x$
repeat
    linearize $r$ near current guess
    new guess is linear LS solution, using linearized $r$
until convergence
Gauss-Newton method, more detail

- linearize $r$ near current iterate $x^{(k)}$:

$$r(x) \approx r(x^{(k)}) + Dr(x^{(k)})(x - x^{(k)})$$

where $Dr$ is the Jacobian: $(Dr)_{ij} = \partial r_i / \partial x_j$

- write linearized approximation as

$$r(x^{(k)}) + Dr(x^{(k)})(x - x^{(k)}) = A^{(k)}x - b^{(k)}$$

$$A^{(k)} = Dr(x^{(k)}), \quad b^{(k)} = Dr(x^{(k)})x^{(k)} - r(x^{(k)})$$

- at $k$th iteration, we approximate NLLS problem by linear LS problem:

$$\|r(x)\|^2 \approx \|A^{(k)}x - b^{(k)}\|^2$$

- next iterate solves this linearized LS problem:

$$x^{(k+1)} = \left( A^{(k)T}A^{(k)} \right)^{-1} A^{(k)T}b^{(k)}$$

- repeat until convergence (which *isn’t* guaranteed)
Gauss-Newton example

- 10 beacons
- + true position \((-3.6, 3.2)\); ◊ initial guess \((1.2, -1.2)\)
- range estimates accurate to ±0.5
Example

NLLS objective $\|r(x)\|^2$ versus $x$:

- for a linear LS problem, objective would be nice quadratic ‘bowl’
- bumps in objective due to strong nonlinearity of $r$
Convergence

objective of Gauss-Newton iterates:

\[ \|r(x)\|_2 \]

- \( x^{(k)} \) converges to (in this case, global) minimum of \( \|r(x)\|^2 \)
- convergence takes only five or so steps
- final estimate is \( \hat{x} = (-3.3, 3.3) \)
- estimation error is \( \|\hat{x} - x\| = 0.31 \), (smaller than range accuracy!)
Convergence

convergence of Gauss-Newton iterates:
Regularized Gauss-Newton

useful variation on Gauss-Newton: add regularization term

$$\| A^{(k)} x - b^{(k)} \|^2 + \mu \| x - x^{(k)} \|^2$$

so that next iterate is not too far from previous one (hence, linearized model still pretty accurate)