Solution via matrix exponential

- matrix exponential
- solving $\dot{x} = Ax$ via matrix exponential
- state transition matrix
- qualitative behavior and stability
Matrix exponential

Define **matrix exponential** as

\[ e^M = I + M + \frac{M^2}{2!} + \cdots \]

- converges for all \( M \in \mathbb{R}^{n \times n} \)
- looks like ordinary power series

\[ e^{at} = 1 + ta + \frac{(ta)^2}{2!} + \cdots \]

with square matrices instead of scalars \ldots
Matrix exponential solution of autonomous LDS

Solution of $\dot{x} = Ax$, with $A \in \mathbb{R}^{n \times n}$ and constant, is

$$x(t) = e^{tA}x(0)$$

The matrix $e^{tA}$ is called the \textit{state transition matrix}, usually written $\Phi(t)$.

Generalizes scalar case: solution of $\dot{x} = ax$, with $a \in \mathbb{R}$ and constant, is

$$x(t) = e^{ta}x(0)$$
Properties of matrix exponential

- matrix exponential is *meant* to look like scalar exponential
- some things you’d guess hold for the matrix exponential (by analogy with the scalar exponential) do in fact hold
- but many things you’d guess are wrong

**example:** you might guess that $e^{A+B} = e^A e^B$, but it's false (in general)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e^A = \begin{bmatrix} 0.54 & 0.84 \\ -0.84 & 0.54 \end{bmatrix}, \quad e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$e^{A+B} = \begin{bmatrix} 0.16 & 1.40 \\ -0.70 & 0.16 \end{bmatrix} \neq e^A e^B = \begin{bmatrix} 0.54 & 1.38 \\ -0.84 & -0.30 \end{bmatrix}$$
Properties of matrix exponential

\[ e^{A+B} = e^A e^B \text{ if } AB = BA \]

i.e., product rule holds when \( A \) and \( B \) commute

thus for \( t, s \in \mathbb{R} \), \( e^{(tA+sA)} = e^{tA} e^{sA} \)

with \( s = -t \) we get
\[
  e^{tA} e^{-tA} = e^{tA-tA} = e^0 = I
\]

so \( e^{tA} \) is nonsingular, with inverse

\[
  \left( e^{tA} \right)^{-1} = e^{-tA}
\]
Example: matrix exponential

Let’s find $e^{tA}$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

The power series gives

$$e^{tA} = I + tA + \frac{t^2 A^2}{2} + \frac{t^3 A^3}{3!} + \cdots$$

$$= I + tA \quad \text{since } A^2 = 0$$

$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

We have $x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0)$
Example: Double integrator

\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x \]
Example: Harmonic oscillator

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \text{ so state transition matrix is} \]

\[
e^{tA} = I + tA + \frac{t^2 A^2}{2} + \frac{t^3 A^3}{3!} + \frac{t^4 A^4}{4!} \cdots
\]

\[
= \left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \cdots\right)I + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)A
\]

\[
= (\cos t)I + (\sin t)A
\]

\[
= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}
\]

a rotation matrix \((-t\ \text{radians})\)

so we have \(x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0)\)
Example 1: Harmonic oscillator

\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x \]
Time transfer property

for $\dot{x} = Ax$ we know

$$x(t) = \Phi(t)x(0) = e^{tA}x(0)$$

**interpretation:** the matrix $e^{tA}$ propagates initial condition into state at time $t$

more generally we have, for *any* $t$ and $\tau$,

$$x(\tau + t) = e^{tA}x(\tau)$$

(to see this, apply result above to $z(t) = x(t + \tau)$)

**interpretation:** the matrix $e^{tA}$ propagates state $t$ seconds forward in time (backward if $t < 0$)
Time transfer property

- recall first order (forward Euler) approximate state update, for small $t$:

$$x(\tau + t) \approx x(\tau) + t\dot{x}(\tau) = (I + tA)x(\tau)$$

- exact solution is

$$x(\tau + t) = e^{tA}x(\tau) = (I + tA + (tA)^2/2! + \cdots)x(\tau)$$

- forward Euler is just first two terms in series
Sampling a continuous-time system

suppose $\dot{x} = Ax$

sample $x$ at times $t_1 \leq t_2 \leq \cdots$: define $z(k) = x(t_k)$

then $z(k + 1) = e^{(t_{k+1} - t_k)A}z(k)$

for uniform sampling $t_{k+1} - t_k = h$, so

$$z(k + 1) = e^{hA}z(k),$$

a discrete-time LDS (called discretized version of continuous-time system)
Piecewise constant system

consider \textit{time-varying} LDS $\dot{x} = A(t)x$, with

$$A(t) = \begin{cases} 
A_0 & 0 \leq t < t_1 \\
A_1 & t_1 \leq t < t_2 \\
\vdots & 
\end{cases}$$

where $0 < t_1 < t_2 < \cdots$ (sometimes called jump linear system)

for $t \in [t_i, t_{i+1}]$ we have

$$x(t) = e^{(t-t_i)A_i} \cdots e^{(t_3-t_2)A_2} e^{(t_2-t_1)A_1} e^{t_1A_0} x(0)$$

(matrix on righthand side is called state transition matrix for system, and denoted $\Phi(t)$)
Stability

we say system $\dot{x} = Ax$ is stable if $e^{tA} \to 0$ as $t \to \infty$

meaning:

- state $x(t)$ converges to 0, as $t \to \infty$, no matter what $x(0)$ is
- all trajectories of $\dot{x} = Ax$ converge to 0 as $t \to \infty$

fact: $\dot{x} = Ax$ is stable if and only if all eigenvalues of $A$ have negative real part:

$$\Re \lambda_i < 0, \quad i = 1, \ldots, n$$
the ‘if’ part is clear since

$$\lim_{t \to \infty} p(t)e^{\lambda t} = 0$$

for any polynomial, if $\Re \lambda < 0$

we'll see the ‘only if’ part next lecture

more generally, $\max_i \Re \lambda_i$ determines the maximum asymptotic logarithmic growth rate of $x(t)$ (or decay, if $< 0$)