Solution via matrix exponential

- matrix exponential
- solving $\dot{x} = Ax$ via matrix exponential
- state transition matrix
- qualitative behavior and stability

Matrix exponential

define **matrix exponential** as

$$e^M = I + M + \frac{M^2}{2!} + \cdots$$

- converges for all $M \in \mathbb{R}^{n \times n}$
- looks like ordinary power series

  $$e^{at} = 1 + ta + \frac{(ta)^2}{2!} + \cdots$$

  with square matrices instead of scalars ...

Matrix exponential solution of autonomous LDS

solution of $\dot{x} = Ax$, with $A \in \mathbb{R}^{n \times n}$ and constant, is

$$x(t) = e^{tA} x(0)$$

the matrix e^{tA} is called the *state transition matrix*, usually written $\Phi(t)$

generalizes scalar case: solution of $\dot{x} = ax$, with $a \in \mathbb{R}$ and constant, is

$$x(t) = e^{ta} x(0)$$

Properties of matrix exponential

- matrix exponential is *meant* to look like scalar exponential
- some things you'd guess hold for the matrix exponential (by analogy with the scalar exponential) do in fact hold
- but **many things you'd guess are wrong**

example: you might guess that $e^{A+B} = e^A e^B$, but it's false (in general)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e^A = \begin{bmatrix} 0.54 & 0.84 \\ -0.84 & 0.54 \end{bmatrix}, \qquad e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$e^{A+B} = \begin{bmatrix} 0.16 & 1.40 \\ -0.70 & 0.16 \end{bmatrix} \neq e^A e^B = \begin{bmatrix} 0.54 & 1.38 \\ -0.84 & -0.30 \end{bmatrix}$$

Properties of matrix exponential

$$e^{A+B} = e^A e^B \text{ if } AB = BA$$

$i.e.$, product rule holds when A and B commute

thus for $t,\ s \in \mathbb{R}$, $e^{(tA+sA)} = e^{tA}e^{sA}$

with $s = -t$ we get

$$e^{tA}e^{-tA} = e^{tA-tA} = e^0 = I$$

so e^{tA} is nonsingular, with inverse

$$\left(e^{tA}\right)^{-1} = e^{-tA}$$

Example: matrix exponential

let's find e^{tA}, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

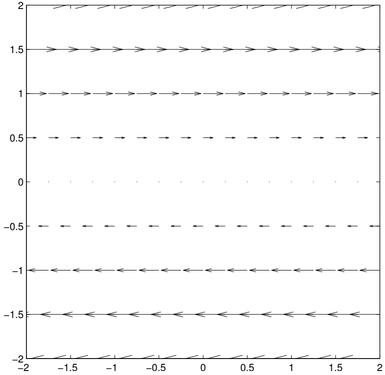
the power series gives

$$\begin{aligned} e^{tA} &= I + tA + \frac{t^2A^2}{2} + \frac{t^3A^3}{3!} + \cdots \\ &= I + tA \qquad \text{since } A^2 = 0 \\ &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \end{aligned}$$

we have $x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0)$

Example: Double integrator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$



Example: Harmonic oscillator

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, so state transition matrix is

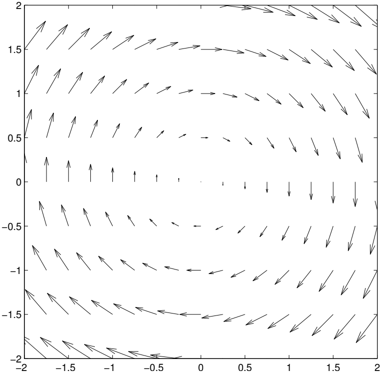
$$\begin{aligned} e^{tA} &= I + tA + \frac{t^2A^2}{2} + \frac{t^3A^3}{3!} + \frac{t^4A^4}{4!} \cdots \\ &= \left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \cdots\right) I + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right) A \\ &= (\cos t)I + (\sin t)A \\ &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \end{aligned}$$

a rotation matrix ($-t$ radians)

so we have $x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0)$

Example 1: Harmonic oscillator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$



Time transfer property

for $\dot{x} = Ax$ we know

$$x(t) = \Phi(t)x(0) = e^{tA}x(0)$$

interpretation: the matrix e^{tA} propagates initial condition into state at time t

more generally we have, for *any* t and τ,

$$x(\tau + t) = e^{tA}x(\tau)$$

(to see this, apply result above to $z(t) = x(t + \tau)$)

interpretation: the matrix e^{tA} propagates state t seconds forward in time (backward if $t < 0$)

Time transfer property

- recall first order (forward Euler) *approximate* state update, for small t:

$$x(\tau + t) \approx x(\tau) + t\dot{x}(\tau) = (I + tA)x(\tau)$$

- *exact* solution is

$$x(\tau + t) = e^{tA}x(\tau) = (I + tA + (tA)^2/2! + \cdots)x(\tau)$$

- forward Euler is just first two terms in series

Sampling a continuous-time system

suppose $\dot{x} = Ax$

sample x at times $t_1 \le t_2 \le \cdots$: define $z(k) = x(t_k)$

then $z(k+1) = e^{(t_{k+1} - t_k)A} z(k)$

for uniform sampling $t_{k+1} - t_k = h$, so

$$z(k+1) = e^{hA} z(k),$$

a discrete-time LDS (called *discretized version* of continuous-time system)

Piecewise constant system

consider *time-varying* LDS $\dot{x} = A(t)x$, with

$$A(t) = \begin{cases} A_0 & 0 \leq t < t_1 \\ A_1 & t_1 \leq t < t_2 \\ \vdots & \end{cases}$$

where $0 < t_1 < t_2 < \cdots$ (sometimes called jump linear system)

for $t \in [t_i, t_{i+1}]$ we have

$$x(t) = e^{(t-t_i)A_i} \cdots e^{(t_3-t_2)A_2} e^{(t_2-t_1)A_1} e^{t_1 A_0} x(0)$$

(matrix on righthand side is called state transition matrix for system, and denoted $\Phi(t)$)

Stability

we say system $\dot{x} = Ax$ is *stable* if $e^{tA} \to 0$ as $t \to \infty$

meaning:

- state $x(t)$ converges to 0, as $t \to \infty$, no matter what $x(0)$ is
- all trajectories of $\dot{x} = Ax$ converge to 0 as $t \to \infty$

fact: $\dot{x} = Ax$ is stable if and only if all eigenvalues of A have negative real part:

$$\Re\lambda_i < 0, \quad i = 1, \ldots, n$$

Stability

the 'if' part is clear since

$$\lim_{t\to\infty} p(t)e^{\lambda t} = 0$$

for any polynomial, if $\Re\lambda < 0$

we'll see the 'only if' part next lecture

more generally, $\mathbf{max}_i \Re\lambda_i$ determines the maximum asymptotic logarithmic growth rate of $x(t)$ (or decay, if < 0)