Solution via Laplace transform and matrix exponential

- Laplace transform
- solving $\dot{x} = Ax$ via Laplace transform
- state transition matrix
- matrix exponential
- qualitative behavior and stability
suppose \( z : \mathbb{R}_+ \rightarrow \mathbb{R}^{p \times q} \)

**Laplace transform:** \( Z = \mathcal{L}(z) \), where \( Z : D \subseteq \mathbb{C} \rightarrow \mathbb{C}^{p \times q} \) is defined by

\[
Z(s) = \int_{0}^{\infty} e^{-st} z(t) \, dt
\]

- integral of matrix is done term-by-term
- convention: upper case denotes Laplace transform
- \( D \) is the *domain* or *region of convergence* of \( Z \)
- \( D \) includes at least \( \{ s \mid \Re s > a \} \), where \( a \) satisfies \( |z_{ij}(t)| \leq \alpha e^{at} \) for \( t \geq 0 \), \( i = 1, \ldots, p, \quad j = 1, \ldots, q \)
Derivative property

\[ \mathcal{L}(\dot{z}) = sZ(s) - z(0) \]

to derive, integrate by parts:

\[
\begin{align*}
\mathcal{L}(\dot{z})(s) &= \int_{0}^{\infty} e^{-st} \dot{z}(t) \, dt \\
&= e^{-st} z(t) \bigg|_{t=0}^{t=\infty} + s \int_{0}^{\infty} e^{-st} z(t) \, dt \\
&= sZ(s) - z(0)
\end{align*}
\]
Laplace transform solution of $\dot{x} = Ax$

consider continuous-time time-invariant (TI) LDS

$$\dot{x} = Ax$$

for $t \geq 0$, where $x(t) \in \mathbb{R}^n$

- take Laplace transform: $sX(s) - x(0) = AX(s)$
- rewrite as $(sI - A)X(s) = x(0)$
- hence $X(s) = (sI - A)^{-1}x(0)$
- take inverse transform

$$x(t) = \mathcal{L}^{-1} \left( (sI - A)^{-1} \right) x(0)$$
(sI − A)^{-1} is called the \textit{resolvent} of A

resolvent defined for \( s \in \mathbb{C} \) except eigenvalues of \( A \), \textit{i.e.}, \( s \) such that \( \det(sI − A) = 0 \)

\( \Phi(t) = \mathcal{L}^{-1} \left( (sI − A)^{-1} \right) \) is called the \textit{state-transition matrix}; it maps the initial state to the state at time \( t \):

\[
x(t) = \Phi(t)x(0)
\]

(in particular, state \( x(t) \) is a linear function of initial state \( x(0) \))
Example 1: Harmonic oscillator

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x
\]
Example 1: Harmonic oscillator

\[ sI - A = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}, \text{ so resolvent is} \]

\[ (sI - A)^{-1} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix} \]

(eigenvalues are \( \pm i \))

state transition matrix is

\[ \Phi(t) = L^{-1} \left( \begin{bmatrix} s/s^2+1 & 1/s^2+1 \\ -1/s^2+1 & s/s^2+1 \end{bmatrix} \right) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \]

a rotation matrix \((-t\ \text{radians})\)

so we have \( x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0) \)
Example 2: Double integrator

\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x \]
Example 2: Double integrator

\[ sI - A = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}, \text{ so resolvent is} \]

\[ (sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & 1/s \end{bmatrix} \]

(eigenvalues are 0, 0)

state transition matrix is

\[ \Phi(t) = \mathcal{L}^{-1} \left( \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & 1/s \end{bmatrix} \right) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \]

so we have \( x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0) \)
Characteristic polynomial

\( \chi(s) = \det(sI - A) \) is called the \textit{characteristic polynomial} of \( A \)

- \( \chi(s) \) is a polynomial of degree \( n \), with leading (\( i.e., \ s^n \)) coefficient one
- roots of \( \chi \) are the eigenvalues of \( A \)
- \( \chi \) has real coefficients, so eigenvalues are either real or occur in conjugate pairs
- there are \( n \) eigenvalues (if we count multiplicity as roots of \( \chi \))
Eigenvalues of $A$ and poles of resolvent

$i, j$ entry of resolvent can be expressed via Cramer’s rule as

\[ (-1)^{i+j} \frac{\det \Delta_{ij}}{\det(sI - A)} \]

where $\Delta_{ij}$ is $sI - A$ with $j$th row and $i$th column deleted.

- $\det \Delta_{ij}$ is a polynomial of degree less than $n$, so $i, j$ entry of resolvent has form $f_{ij}(s) / \chi(s)$ where $f_{ij}$ is polynomial with degree less than $n$

- poles of entries of resolvent must be eigenvalues of $A$

- but not all eigenvalues of $A$ show up as poles of each entry (when there are cancellations between $\det \Delta_{ij}$ and $\chi(s)$)
Matrix exponential

define **matrix exponential** as

\[
\begin{align*}
    e^M &= I + M + \frac{M^2}{2!} + \cdots \\
    e^{at} &= 1 + ta + \frac{(ta)^2}{2!} + \cdots
\end{align*}
\]

- converges for all \( M \in \mathbb{R}^{n \times n} \)
- looks like ordinary power series

with square matrices instead of scalars . . .
Matrix exponential

\[(I - C)^{-1} = I + C + C^2 + C^3 + \cdots \quad \text{(if series converges)}\]

▷ series expansion of resolvent:

\[ (sI - A)^{-1} = (1/s)(I - A/s)^{-1} = I + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots \]

(valid for \(|s|\) large enough) so

\[ \Phi(t) = \mathcal{L}^{-1} \left( (sI - A)^{-1} \right) = I + tA + \frac{(tA)^2}{2!} + \cdots \]

▷ with this definition, state-transition matrix is

\[ \Phi(t) = \mathcal{L}^{-1} \left( (sI - A)^{-1} \right) = e^{tA} \]
solution of $\dot{x} = Ax$, with $A \in \mathbb{R}^{n \times n}$ and constant, is

$$x(t) = e^{tA}x(0)$$

generalizes scalar case: solution of $\dot{x} = ax$, with $a \in \mathbb{R}$ and constant, is

$$x(t) = e^{ta}x(0)$$
**Example: matrix exponential**

Let's find $e^A$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

We already found

$$e^{tA} = L^{-1}(sI - A)^{-1} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

So, plugging in $t = 1$, we get $e^A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Let's check power series:

$$e^A = I + A + \frac{A^2}{2!} + \cdots = I + A$$

Since $A^2 = A^3 = \cdots = 0$.
Properties of matrix exponential

- matrix exponential is *meant* to look like scalar exponential
- some things you’d guess hold for the matrix exponential (by analogy with the scalar exponential) do in fact hold
- but *many things you’d guess are wrong*

**example:** you might guess that $e^{A+B} = e^A e^B$, but it’s false (in general)

\[
A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

\[
e^A = \begin{bmatrix} 0.54 & 0.84 \\ -0.84 & 0.54 \end{bmatrix}, \quad e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}
\]

\[
e^{A+B} = \begin{bmatrix} 0.16 & 1.40 \\ -0.70 & 0.16 \end{bmatrix} \neq e^A e^B = \begin{bmatrix} 0.54 & 1.38 \\ -0.84 & -0.30 \end{bmatrix}
\]
Properties of matrix exponential

\[ e^{A+B} = e^A e^B \text{ if } AB = BA \]

\text{i.e., product rule holds when } A \text{ and } B \text{ commute}

thus for \( t, s \in \mathbb{R} \),
\[ e^{(tA+sA)} = e^{tA} e^{sA} \]

with \( s = -t \) we get
\[ e^{tA} e^{-tA} = e^{tA-tA} = e^0 = I \]

so \( e^{tA} \) is nonsingular, with inverse
\[ \left(e^{tA}\right)^{-1} = e^{-tA} \]
**Time transfer property**

for $\dot{x} = Ax$ we know

$$x(t) = \Phi(t)x(0) = e^{tA}x(0)$$

**interpretation:** the matrix $e^{tA}$ propagates initial condition into state at time $t$

more generally we have, for *any* $t$ and $\tau$, 

$$x(\tau + t) = e^{tA}x(\tau)$$

(to see this, apply result above to $z(t) = x(t + \tau)$)

**interpretation:** the matrix $e^{tA}$ propagates state $t$ seconds forward in time (backward if $t < 0$)
Time transfer property

- recall first order (forward Euler) approximate state update, for small $t$:
  \[ x(\tau + t) \approx x(\tau) + t\dot{x}(\tau) = (I + tA)x(\tau) \]

- exact solution is
  \[ x(\tau + t) = e^{tA}x(\tau) = (I + tA + (tA)^2/2! + \cdots)x(\tau) \]

- forward Euler is just first two terms in series
Sampling a continuous-time system

suppose $\dot{x} = Ax$

sample $x$ at times $t_1 \leq t_2 \leq \cdots$: define $z(k) = x(t_k)$

then $z(k + 1) = e^{(t_{k+1} - t_k)A}z(k)$

for uniform sampling $t_{k+1} - t_k = h$, so

$$z(k + 1) = e^{hA}z(k),$$

a discrete-time LDS (called discretized version of continuous-time system)
Piecewise constant system

consider *time-varying* LDS \( \dot{x} = A(t)x \), with

\[
A(t) = \begin{cases} 
    A_0 & 0 \leq t < t_1 \\
    A_1 & t_1 \leq t < t_2 \\
    \vdots 
\end{cases}
\]

where \( 0 < t_1 < t_2 < \cdots \) (sometimes called jump linear system)

for \( t \in [t_i, t_{i+1}] \) we have

\[
x(t) = e^{(t-t_i)A_i} \cdots e^{(t_3-t_2)A_2} e^{(t_2-t_1)A_1} e^{t_1A_0} x(0)
\]

(matrix on righthand side is called state transition matrix for system, and denoted \( \Phi(t) \))
Qualitative behavior of $x(t)$

Suppose $\dot{x} = Ax$, $x(t) \in \mathbb{R}^n$ then $x(t) = e^{tA}x(0)$; $X(s) = (sI - A)^{-1}x(0)$

$i$th component $X_i(s)$ has form

$$X_i(s) = \frac{a_i(s)}{\mathcal{X}(s)}$$

where $a_i$ is a polynomial of degree $< n$

thus the poles of $X_i$ are all eigenvalues of $A$ (but not necessarily the other way around)
Qualitative behavior of \( x(t) \)

first assume eigenvalues \( \lambda_i \) are distinct, so \( X_i(s) \) cannot have repeated poles

then \( x_i(t) \) has form

\[
x_i(t) = \sum_{j=1}^{n} \beta_{ij} e^{\lambda_j t}
\]

where \( \beta_{ij} \) depend on \( x(0) \) (linearly)

eigenvalues determine (possible) qualitative behavior of \( x \):

- eigenvalues give exponents that can occur in exponentials

- real eigenvalue \( \lambda \) corresponds to an exponentially decaying or growing term \( e^{\lambda t} \) in solution

- complex eigenvalue \( \lambda = \sigma + i\omega \) corresponds to decaying or growing sinusoidal term \( e^{\sigma t} \cos(\omega t + \phi) \) in solution
Qualitative behavior of $x(t)$

- $\Re \lambda_j$ gives exponential growth rate (if $> 0$), or exponential decay rate (if $< 0$) of term

- $\Im \lambda_j$ gives frequency of oscillatory term (if $\neq 0$)
Repeated eigenvalues

now suppose $A$ has repeated eigenvalues, so $X_i$ can have repeated poles

express eigenvalues as $\lambda_1, \ldots, \lambda_r$ (distinct) with multiplicities $n_1, \ldots, n_r$, respectively ($n_1 + \cdots + n_r = n$)

then $x_i(t)$ has form

$$x_i(t) = \sum_{j=1}^{r} p_{ij}(t)e^{\lambda_j t}$$

where $p_{ij}(t)$ is a polynomial of degree $< n_j$ (that depends linearly on $x(0)$)
Stability

we say system $\dot{x} = Ax$ is stable if $e^{tA} \to 0$ as $t \to \infty$

meaning:

- state $x(t)$ converges to 0, as $t \to \infty$, no matter what $x(0)$ is
- all trajectories of $\dot{x} = Ax$ converge to 0 as $t \to \infty$

fact: $\dot{x} = Ax$ is stable if and only if all eigenvalues of $A$ have negative real part:

$$\Re \lambda_i < 0, \quad i = 1, \ldots, n$$
Stability

the ‘if’ part is clear since

\[ \lim_{t \to \infty} p(t)e^{\lambda t} = 0 \]

for any polynomial, if \( \Re \lambda < 0 \)

we'll see the ‘only if’ part next lecture

more generally, \( \max_i \Re \lambda_i \) determines the maximum asymptotic logarithmic growth rate of \( x(t) \) (or decay, if \( < 0 \))