

## Solution via matrix exponential

- ▶ matrix exponential
- ▶ solving  $\dot{x} = Ax$  via matrix exponential
- ▶ state transition matrix
- ▶ qualitative behavior and stability

## Matrix exponential

define **matrix exponential** as

$$e^M = I + M + \frac{M^2}{2!} + \dots$$

- ▶ converges for all  $M \in \mathbb{R}^{n \times n}$
- ▶ looks like ordinary power series

$$e^{at} = 1 + ta + \frac{(ta)^2}{2!} + \dots$$

with square matrices instead of scalars ...

## Matrix exponential solution of autonomous LDS

solution of  $\dot{x} = Ax$ , with  $A \in \mathbb{R}^{n \times n}$  and constant, is

$$x(t) = e^{tA} x(0)$$

the matrix  $e^{tA}$  is called the *state transition matrix*, usually written  $\Phi(t)$

generalizes scalar case: solution of  $\dot{x} = ax$ , with  $a \in \mathbb{R}$  and constant, is

$$x(t) = e^{ta} x(0)$$

## Properties of matrix exponential

- ▶ matrix exponential is *meant* to look like scalar exponential
- ▶ some things you'd guess hold for the matrix exponential (by analogy with the scalar exponential) do in fact hold
- ▶ but **many things you'd guess are wrong**

**example:** you might guess that  $e^{A+B} = e^A e^B$ , but it's false (in general)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e^A = \begin{bmatrix} 0.54 & 0.84 \\ -0.84 & 0.54 \end{bmatrix}, \quad e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$e^{A+B} = \begin{bmatrix} 0.16 & 1.40 \\ -0.70 & 0.16 \end{bmatrix} \neq e^A e^B = \begin{bmatrix} 0.54 & 1.38 \\ -0.84 & -0.30 \end{bmatrix}$$

## Properties of matrix exponential

$$e^{A+B} = e^A e^B \text{ if } AB = BA$$

*i.e.*, product rule holds when  $A$  and  $B$  commute

thus for  $t, s \in \mathbb{R}$ ,  $e^{(tA+sA)} = e^{tA} e^{sA}$

with  $s = -t$  we get

$$e^{tA} e^{-tA} = e^{tA-tA} = e^0 = I$$

so  $e^{tA}$  is nonsingular, with inverse

$$(e^{tA})^{-1} = e^{-tA}$$

## Example: matrix exponential

let's find  $e^{tA}$ , where  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

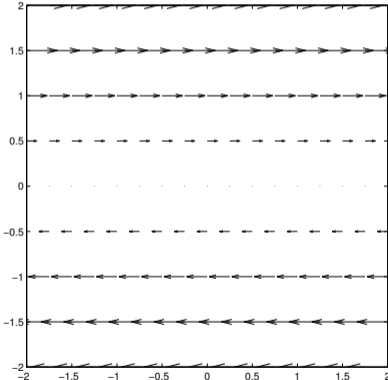
the power series gives

$$\begin{aligned} e^{tA} &= I + tA + \frac{t^2 A^2}{2} + \frac{t^3 A^3}{3!} + \dots \\ &= I + tA \quad \text{since } A^2 = 0 \\ &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \end{aligned}$$

we have  $x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0)$

# Example: Double integrator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$



## Example: Harmonic oscillator

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ , so state transition matrix is

$$\begin{aligned} e^{tA} &= I + tA + \frac{t^2 A^2}{2} + \frac{t^3 A^3}{3!} + \frac{t^4 A^4}{4!} \dots \\ &= \left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \dots\right) I + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right) A \\ &= (\cos t)I + (\sin t)A \\ &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \end{aligned}$$

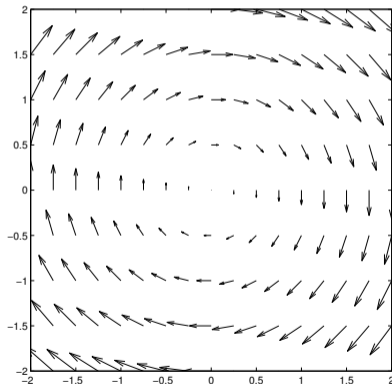
a rotation matrix ( $-t$  radians)

so we have  $x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0)$



## Example 1: Harmonic oscillator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$



## Time transfer property

for  $\dot{x} = Ax$  we know

$$x(t) = \Phi(t)x(0) = e^{tA}x(0)$$

**interpretation:** the matrix  $e^{tA}$  propagates initial condition into state at time  $t$

more generally we have, for *any*  $t$  and  $\tau$ ,

$$x(\tau + t) = e^{tA}x(\tau)$$

(to see this, apply result above to  $z(t) = x(t + \tau)$ )

**interpretation:** the matrix  $e^{tA}$  propagates state  $t$  seconds forward in time (backward if  $t < 0$ )

## Time transfer property

- ▶ recall first order (forward Euler) *approximate* state update, for small  $t$ :

$$x(\tau + t) \approx x(\tau) + t\dot{x}(\tau) = (I + tA)x(\tau)$$

- ▶ *exact* solution is

$$x(\tau + t) = e^{tA}x(\tau) = (I + tA + (tA)^2/2! + \dots)x(\tau)$$

- ▶ forward Euler is just first two terms in series

## Sampling a continuous-time system

suppose  $\dot{x} = Ax$

sample  $x$  at times  $t_1 \leq t_2 \leq \dots$ : define  $z(k) = x(t_k)$

then  $z(k+1) = e^{(t_{k+1}-t_k)A}z(k)$

for uniform sampling  $t_{k+1} - t_k = h$ , so

$$z(k+1) = e^{hA}z(k),$$

a discrete-time LDS (called *discretized version* of continuous-time system)

## Piecewise constant system

consider *time-varying* LDS  $\dot{x} = A(t)x$ , with

$$A(t) = \begin{cases} A_0 & 0 \leq t < t_1 \\ A_1 & t_1 \leq t < t_2 \\ \vdots & \end{cases}$$

where  $0 < t_1 < t_2 < \dots$  (sometimes called jump linear system)

for  $t \in [t_i, t_{i+1}]$  we have

$$x(t) = e^{(t-t_i)A_i} \dots e^{(t_3-t_2)A_2} e^{(t_2-t_1)A_1} e^{t_1 A_0} x(0)$$

(matrix on righthand side is called state transition matrix for system, and denoted  $\Phi(t)$ )

## Stability

we say system  $\dot{x} = Ax$  is *stable* if  $e^{tA} \rightarrow 0$  as  $t \rightarrow \infty$

**meaning:**

- ▶ state  $x(t)$  converges to 0, as  $t \rightarrow \infty$ , no matter what  $x(0)$  is
- ▶ all trajectories of  $\dot{x} = Ax$  converge to 0 as  $t \rightarrow \infty$

**fact:**  $\dot{x} = Ax$  is stable if and only if all eigenvalues of  $A$  have negative real part:

$$\Re \lambda_i < 0, \quad i = 1, \dots, n$$

## Stability

the 'if' part is clear since

$$\lim_{t \rightarrow \infty} p(t)e^{\lambda t} = 0$$

for any polynomial, if  $\Re \lambda < 0$

we'll see the 'only if' part next lecture

more generally,  $\max_i \Re \lambda_i$  determines the maximum asymptotic logarithmic growth rate of  $x(t)$  (or decay, if  $< 0$ )