Solution via Laplace transform and matrix exponential

- Laplace transform
- solving $\dot{x} = Ax$ via Laplace transform
- state transition matrix
- matrix exponential
- qualitative behavior and stability
suppose $z : \mathbb{R}_+ \rightarrow \mathbb{R}^{p \times q}$

**Laplace transform**: $Z = \mathcal{L}(z)$, where $Z : D \subseteq \mathbb{C} \rightarrow \mathbb{C}^{p \times q}$ is defined by

$$Z(s) = \int_{0}^{\infty} e^{-st} z(t) \, dt$$

- integral of matrix is done term-by-term
- convention: upper case denotes Laplace transform
- $D$ is the *domain* or *region of convergence* of $Z$
- $D$ includes at least $\{s \mid \Re{s} > a\}$, where $a$ satisfies $|z_{ij}(t)| \leq \alpha e^{at}$ for $t \geq 0$, $i = 1, \ldots, p$, $j = 1, \ldots, q$
Derivative property

\[ \mathcal{L}(\dot{z}) = sZ(s) - z(0) \]

To derive, integrate by parts:

\[
\mathcal{L}(\dot{z})(s) = \int_0^\infty e^{-st} \dot{z}(t) \, dt \\
= e^{-st} z(t) \bigg|_{t=0}^{t=\infty} + s \int_0^\infty e^{-st} z(t) \, dt \\
= sZ(s) - z(0)
\]
Laplace transform solution of $\dot{x} = Ax$

Consider continuous-time time-invariant (TI) LDS

$$\dot{x} = Ax$$

for $t \geq 0$, where $x(t) \in \mathbb{R}^n$

- Take Laplace transform: $sX(s) - x(0) = AX(s)$
- Rewrite as $(sI - A)X(s) = x(0)$
- Hence $X(s) = (sI - A)^{-1}x(0)$
- Take inverse transform

$$x(t) = \mathcal{L}^{-1} \left( (sI - A)^{-1} \right) x(0)$$
Resolvent and state transition matrix

- \((sI - A)^{-1}\) is called the *resolvent* of \(A\)

- Resolvent defined for \(s \in \mathbb{C}\) except eigenvalues of \(A\), i.e., \(s\) such that \(\det(sI - A) = 0\)

- \(\Phi(t) = \mathcal{L}^{-1} ((sI - A)^{-1})\) is called the *state-transition matrix*; it maps the initial state to the state at time \(t\):

  \[
x(t) = \Phi(t)x(0)
  \]

  (in particular, state \(x(t)\) is a linear function of initial state \(x(0)\))
Example 1: Harmonic oscillator

\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x \]
**Example 1: Harmonic oscillator**

\[ sI - A = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}, \text{ so resolvent is} \]

\[ (sI - A)^{-1} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ -1 & \frac{s}{s^2+1} \end{bmatrix} \]

(eigenvalues are \( \pm i \))

state transition matrix is

\[ \Phi(t) = \mathcal{L}^{-1} \left( \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ -1 & \frac{s}{s^2+1} \end{bmatrix} \right) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \]

a rotation matrix (\(-t\) radians)

so we have \( x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0) \)
Example 2: Double integrator

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x
\]
Example 2: Double integrator

\[ sI - A = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}, \text{ so resolvent is} \]

\[ (sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix} \]

(eigenvalues are 0, 0)

state transition matrix is

\[ \Phi(t) = \mathcal{L}^{-1} \left( \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix} \right) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \]

so we have \[ x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0) \]
Characteristic polynomial

\[ \chi(s) = \det(sI - A) \] is called the \textit{characteristic polynomial} of \( A \)

\begin{itemize}
  \item \( \chi(s) \) is a polynomial of degree \( n \), with leading (\textit{i.e.}, \( s^n \)) coefficient one
  \item roots of \( \chi \) are the eigenvalues of \( A \)
  \item \( \chi \) has real coefficients, so eigenvalues are either real or occur in conjugate pairs
  \item there are \( n \) eigenvalues (if we count multiplicity as roots of \( \chi \))
\end{itemize}
Eigenvalues of $A$ and poles of resolvent

$i, j$ entry of resolvent can be expressed via Cramer’s rule as

$$(-1)^{i+j} \frac{\det \Delta_{ij}}{\det(sI - A)}$$

where $\Delta_{ij}$ is $sI - A$ with $j$th row and $i$th column deleted

- $\det \Delta_{ij}$ is a polynomial of degree less than $n$, so $i, j$ entry of resolvent has form $f_{ij}(s)/\chi(s)$ where $f_{ij}$ is polynomial with degree less than $n$

- poles of entries of resolvent must be eigenvalues of $A$

- but not all eigenvalues of $A$ show up as poles of each entry (when there are cancellations between $\det \Delta_{ij}$ and $\chi(s)$)
Matrix exponential

Define \textbf{matrix exponential} as

\[ e^M = I + M + \frac{M^2}{2!} + \cdots \]

- converges for all \( M \in \mathbb{R}^{n \times n} \)
- looks like ordinary power series

\[ e^{at} = 1 + ta + \frac{(ta)^2}{2!} + \cdots \]

with square matrices instead of scalars \ldots
Matrix exponential

\[(I - C)^{-1} = I + C + C^2 + C^3 + \cdots \text{ (if series converges)}\]

- series expansion of resolvent:

\[(sI - A)^{-1} = (1/s)(I - A/s)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots\]

(valid for \(|s|\) large enough) so

\[\Phi(t) = \mathcal{L}^{-1} \left((sI - A)^{-1}\right) = I + tA + \frac{(tA)^2}{2!} + \cdots\]

- with this definition, state-transition matrix is

\[\Phi(t) = \mathcal{L}^{-1} \left((sI - A)^{-1}\right) = e^{tA}\]
solution of $\dot{x} = Ax$, with $A \in \mathbb{R}^{n \times n}$ and constant, is

$$x(t) = e^{tA}x(0)$$

generalizes scalar case: solution of $\dot{x} = ax$, with $a \in \mathbb{R}$ and constant, is

$$x(t) = e^{ta}x(0)$$
Properties of matrix exponential

- matrix exponential is *meant* to look like scalar exponential
- some things you’d guess hold for the matrix exponential (by analogy with the scalar exponential) do in fact hold
- but many things you’d guess are wrong

**example:** you might guess that $e^{A+B} = e^A e^B$, but it’s false (in general)

\[
A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

\[
e^A = \begin{bmatrix} 0.54 & 0.84 \\ -0.84 & 0.54 \end{bmatrix}, \quad e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}
\]

\[
e^{A+B} = \begin{bmatrix} 0.16 & 1.40 \\ -0.70 & 0.16 \end{bmatrix} \neq e^A e^B = \begin{bmatrix} 0.54 & 1.38 \\ -0.84 & -0.30 \end{bmatrix}
\]
Properties of matrix exponential

\[ e^{A+B} = e^A e^B \text{ if } AB = BA \]

i.e., product rule holds when \( A \) and \( B \) commute

thus for \( t, \ s \in \mathbb{R} \), \( e^{(tA+sA)} = e^{tA} e^{sA} \)

with \( s = -t \) we get

\[ e^{tA} e^{-tA} = e^{tA-tA} = e^0 = I \]

so \( e^{tA} \) is nonsingular, with inverse

\[ (e^{tA})^{-1} = e^{-tA} \]
Example: matrix exponential

let’s find $e^A$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

we already found

$$e^{tA} = \mathcal{L}^{-1}(sI - A)^{-1} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

so, plugging in $t = 1$, we get $e^A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

let’s check power series:

$$e^A = I + A + \frac{A^2}{2!} + \cdots = I + A$$

since $A^2 = A^3 = \cdots = 0$
Time transfer property

for \( \dot{x} = Ax \) we know

\[
x(t) = \Phi(t)x(0) = e^{tA}x(0)
\]

**interpretation:** the matrix \( e^{tA} \) propagates initial condition into state at time \( t \)

more generally we have, for *any* \( t \) and \( \tau \),

\[
x(\tau + t) = e^{tA}x(\tau)
\]

(to see this, apply result above to \( z(t) = x(t + \tau) \))

**interpretation:** the matrix \( e^{tA} \) propagates state \( t \) seconds forward in time (backward if \( t < 0 \))
Time transfer property

- recall first order (forward Euler) approximate state update, for small $t$:

\[ x(\tau + t) \approx x(\tau) + t\dot{x}(\tau) = (I + tA)x(\tau) \]

- exact solution is

\[ x(\tau + t) = e^{tA}x(\tau) = (I + tA + (tA)^2/2! + \cdots)x(\tau) \]

- forward Euler is just first two terms in series
Sampling a continuous-time system

Suppose \( \dot{x} = Ax \)

Sample \( x \) at times \( t_1 \leq t_2 \leq \cdots \): define \( z(k) = x(t_k) \)

Then \( z(k+1) = e^{(t_{k+1} - t_k)A} z(k) \)

For uniform sampling \( t_{k+1} - t_k = h \), so

\[
z(k+1) = e^{hA} z(k),
\]

A discrete-time LDS (called discretized version of continuous-time system)
Piecewise constant system

consider time-varying LDS $\dot{x} = A(t)x$, with

$$A(t) = \begin{cases} A_0 & 0 \leq t < t_1 \\ A_1 & t_1 \leq t < t_2 \\ \vdots \\ \end{cases}$$

where $0 < t_1 < t_2 < \cdots$ (sometimes called jump linear system)

for $t \in [t_i, t_{i+1}]$ we have

$$x(t) = e^{(t-t_i)A_i} \cdots e^{(t_3-t_2)A_2} e^{(t_2-t_1)A_1} e^{t_1A_0} x(0)$$

(matrix on righthand side is called state transition matrix for system, and denoted $\Phi(t)$)
Qualitative behavior of $x(t)$

suppose $\dot{x} = Ax$, $x(t) \in \mathbb{R}^n$
then $x(t) = e^{tA}x(0)$; $X(s) = (sI - A)^{-1}x(0)$
i$\text{th component } X_i(s) \text{ has form}$

$$X_i(s) = \frac{a_i(s)}{\lambda'(s)}$$

where $a_i$ is a polynomial of degree $< n$
thus the poles of $X_i$ are all eigenvalues of $A$ (but not necessarily the other way around)
Qualitative behavior of $x(t)$

first assume eigenvalues $\lambda_i$ are distinct, so $X_i(s)$ cannot have repeated poles
then $x_i(t)$ has form

$$x_i(t) = \sum_{j=1}^{n} \beta_{ij} e^{\lambda_j t}$$

where $\beta_{ij}$ depend on $x(0)$ (linearly)
eigenvalues determine (possible) qualitative behavior of $x$:

- eigenvalues give exponents that can occur in exponentials
- real eigenvalue $\lambda$ corresponds to an exponentially decaying or growing term $e^{\lambda t}$ in solution
- complex eigenvalue $\lambda = \sigma + i\omega$ corresponds to decaying or growing sinusoidal term $e^{\sigma t} \cos(\omega t + \phi)$ in solution
Qualitative behavior of $x(t)$

- $\Re \lambda_j$ gives exponential growth rate (if $>0$), or exponential decay rate (if $<0$) of term

- $\Im \lambda_j$ gives frequency of oscillatory term (if $\neq 0$)
Repeated eigenvalues

now suppose \( A \) has repeated eigenvalues, so \( X_i \) can have repeated poles

express eigenvalues as \( \lambda_1, \ldots, \lambda_r \) (distinct) with multiplicities \( n_1, \ldots, n_r \), respectively \( (n_1 + \cdots + n_r = n) \)

then \( x_i(t) \) has form

\[
x_i(t) = \sum_{j=1}^{r} p_{ij}(t)e^{\lambda_j t}
\]

where \( p_{ij}(t) \) is a polynomial of degree \( < n_j \) (that depends linearly on \( x(0) \))
Stability

we say system $\dot{x} = Ax$ is stable if $e^{tA} \to 0$ as $t \to \infty$

meaning:

- state $x(t)$ converges to 0, as $t \to \infty$, no matter what $x(0)$ is
- all trajectories of $\dot{x} = Ax$ converge to 0 as $t \to \infty$

fact: $\dot{x} = Ax$ is stable if and only if all eigenvalues of $A$ have negative real part:

$$\Re \lambda_i < 0, \quad i = 1, \ldots, n$$
the ‘if’ part is clear since

$$\lim_{t \to \infty} p(t)e^{\lambda t} = 0$$

for any polynomial, if $\Re \lambda < 0$

we’ll see the ‘only if’ part next lecture

more generally, $\max_i \Re \lambda_i$ determines the maximum asymptotic logarithmic growth rate of $x(t)$ (or decay, if $< 0$)