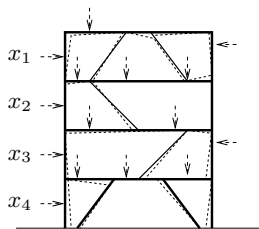


Engineering examples

Linear elastic structure

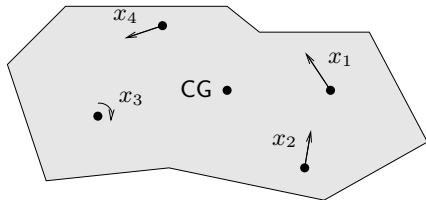
- ▶ x_j is external force applied at some node, in some fixed direction
- ▶ y_i is (small) deflection of some node, in some fixed direction



(provided x , y are small) we have $y \approx Ax$

- ▶ A is called the *compliance matrix*
- ▶ a_{ij} gives deflection i per unit force at j (in m/N)

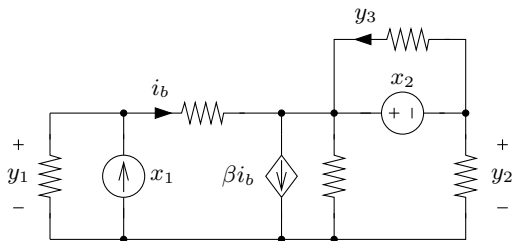
Total force/torque on rigid body



- ▶ x_j is external force/torque applied at some point/direction/axis
- ▶ $y \in \mathbb{R}^6$ is resulting total force & torque on body
(y_1, y_2, y_3 are \mathbf{x} -, \mathbf{y} -, \mathbf{z} - components of total force,
 y_4, y_5, y_6 are \mathbf{x} -, \mathbf{y} -, \mathbf{z} - components of total torque)
- ▶ we have $y = Ax$
- ▶ A depends on geometry
(of applied forces and torques with respect to center of gravity CG)
- ▶ j th column gives resulting force & torque for unit force/torque j

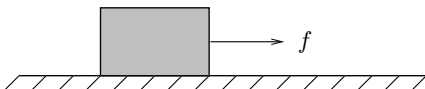
Linear static circuit

interconnection of resistors, linear dependent (controlled) sources, and independent sources



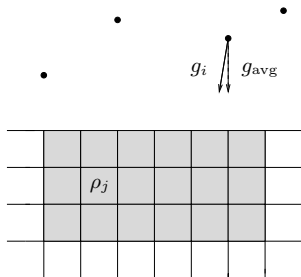
- ▶ x_j is value of independent source j
- ▶ y_i is some circuit variable (voltage, current)
- ▶ we have $y = Ax$
- ▶ if x_j are currents and y_i are voltages, A is called the *impedance* or *resistance* matrix

Final position/velocity of mass due to applied forces



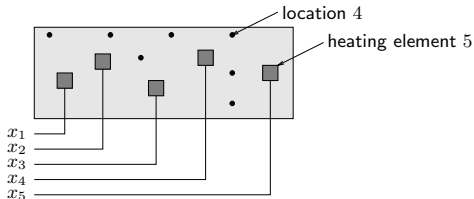
- ▶ unit mass, zero position/velocity at $t = 0$, subject to force $f(t)$ for $0 \leq t \leq n$
- ▶ $f(t) = x_j$ for $j - 1 \leq t < j$, $j = 1, \dots, n$
(x is the sequence of applied forces, constant in each interval)
- ▶ y_1, y_2 are final position and velocity (*i.e.*, at $t = n$)
- ▶ we have $y = Ax$
- ▶ a_{1j} gives influence of applied force during $j - 1 \leq t < j$ on final position
- ▶ a_{2j} gives influence of applied force during $j - 1 \leq t < j$ on final velocity

Gravimeter prospecting



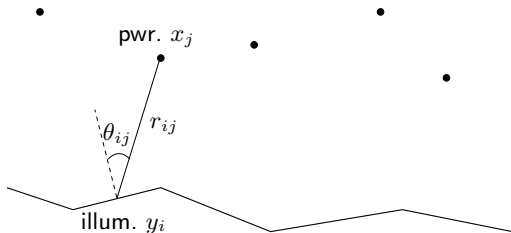
- ▶ $x_j = \rho_j - \rho_{avg}$ is (excess) mass density of earth in voxel j ;
- ▶ y_i is measured *gravity anomaly* at location i , *i.e.*, some component (typically vertical) of $g_i - g_{avg}$
- ▶ $y = Ax$, where A comes from physics and geometry
- ▶ j th column of A shows sensor readings caused by unit density anomaly at voxel j
- ▶ i th row of A shows sensitivity pattern of sensor i

Thermal system



- ▶ x_j is power of j th heating element or heat source
- ▶ y_i is change in steady-state temperature at location i
- ▶ thermal transport via conduction
- ▶ $y = Ax$
- ▶ a_{ij} gives influence of heater j at location i (in $^{\circ}\text{C}/\text{W}$)
- ▶ j th column of A gives pattern of steady-state temperature rise due to 1W at heater j
- ▶ i th row shows how heaters affect location i

Illumination with multiple lamps



- ▶ n lamps illuminating m (small, flat) patches, no shadows
- ▶ x_j is power of j th lamp; y_i is illumination level of patch i
- ▶ $y = Ax$, where $a_{ij} = r_{ij}^{-2} \max\{\cos \theta_{ij}, 0\}$
($\cos \theta_{ij} < 0$ means patch i is shaded from lamp j)
- ▶ j th column of A shows illumination pattern from lamp j

Signal and interference power in wireless system

- ▶ n transmitter/receiver pairs
- ▶ transmitter j transmits to receiver j (and, inadvertently, to the other receivers)
- ▶ p_j is power of j th transmitter
- ▶ s_i is received signal power of i th receiver
- ▶ z_i is received interference power of i th receiver
- ▶ G_{ij} is path gain from transmitter j to receiver i
- ▶ we have $s = Ap$, $z = Bp$, where

$$a_{ij} = \begin{cases} G_{ii} & i = j \\ 0 & i \neq j \end{cases} \quad b_{ij} = \begin{cases} 0 & i = j \\ G_{ij} & i \neq j \end{cases}$$

- ▶ A is diagonal; B has zero diagonal (ideally, A is 'large', B is 'small')

Cost of production

production *inputs* (materials, parts, labor, ...) are combined to make a number of *products*

- ▶ x_j is price per unit of production input j
- ▶ a_{ij} is units of production input j required to manufacture one unit of product i
- ▶ y_i is production cost per unit of product i
- ▶ we have $y = Ax$
- ▶ i th row of A is *bill of materials* for unit of product i

Cost of production

production inputs needed

- ▶ q_i is quantity of product i to be produced
- ▶ r_j is total quantity of production input j needed
- ▶ we have $r = A^T q$

total production cost is

$$r^T x = (A^T q)^T x = q^T Ax$$

Network traffic and flows

- ▶ n flows with rates f_1, \dots, f_n pass from their source nodes to their destination nodes over fixed routes in a network
- ▶ t_i , traffic on link i , is sum of rates of flows passing through it
- ▶ flow routes given by *flow-link incidence matrix*

$$A_{ij} = \begin{cases} 1 & \text{flow } j \text{ goes over link } i \\ 0 & \text{otherwise} \end{cases}$$

- ▶ traffic and flow rates related by $t = Af$

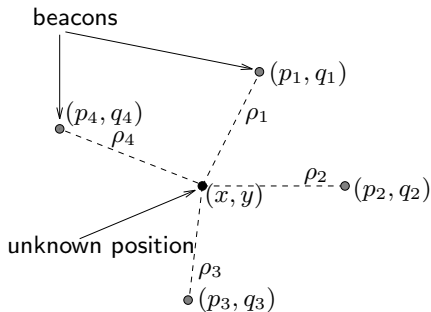
Network traffic and flows

link delays and flow latency

- ▶ let d_1, \dots, d_m be link delays, and l_1, \dots, l_n be latency (total travel time) of flows
- ▶ $l = A^T d$
- ▶ $f^T l = f^T A^T d = (Af)^T d = t^T d$, total # of packets in network

Navigation by range measurement

- ▶ (x, y) unknown coordinates in plane
- ▶ (p_i, q_i) known coordinates of beacons for $i = 1, 2, 3, 4$
- ▶ ρ_i measured (known) distance or range from beacon i



Navigation by range measurement

- ▶ $\rho \in \mathbb{R}^4$ is a nonlinear function of $(x, y) \in \mathbb{R}^2$

$$\rho_i(x, y) = \sqrt{(x - p_i)^2 + (y - q_i)^2}$$

- ▶ linearize around (x_0, y_0) : $\delta\rho \approx A \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$, where

$$a_{i1} = \frac{(x_0 - p_i)}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}, \quad a_{i2} = \frac{(y_0 - q_i)}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}$$

- ▶ i th row of A shows (approximate) change in i th range measurement for (small) shift in (x, y) from (x_0, y_0)
- ▶ first column of A shows sensitivity of range measurements to (small) change in x from x_0
- ▶ obvious application: (x_0, y_0) is last navigation fix; (x, y) is current position, a short time later