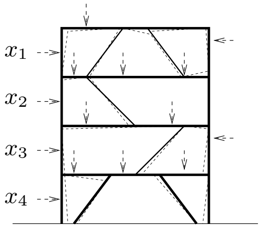
Example: Linear Models

Linear elastic structure

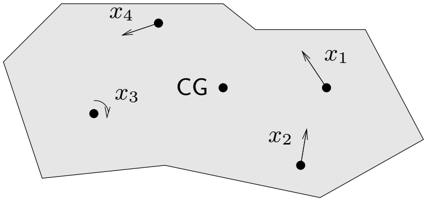
- x_j is external force applied at some node, in some fixed direction
- y_i is (small) deflection of some node, in some fixed direction



(provided x, y are small) we have $y \approx Ax$

- A is called the *compliance matrix*
- a_{ij} gives deflection i per unit force at j (in m/N)

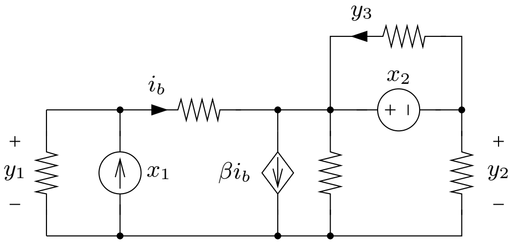
Total force/torque on rigid body



- x_j is external force/torque applied at some point/direction/axis
- $y \in \mathbb{R}^6$ is resulting total force & torque on body
  ($y_1,\ y_2,\ y_3$ are $\mathbf{x}$-, $\mathbf{y}$-, $\mathbf{z}$- components of total force,
  $y_4,\ y_5,\ y_6$ are $\mathbf{x}$-, $\mathbf{y}$-, $\mathbf{z}$- components of total torque)
- we have $y = Ax$
- A depends on geometry
  (of applied forces and torques with respect to center of gravity CG)
- jth column gives resulting force & torque for unit force/torque j

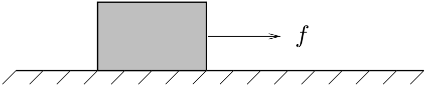
Linear static circuit

interconnection of resistors, linear dependent (controlled) sources, and independent sources



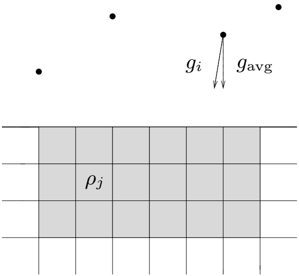
- x_j is value of independent source j
- y_i is some circuit variable (voltage, current)
- we have $y = Ax$
- if x_j are currents and y_i are voltages, A is called the *impedance* or *resistance* matrix

Final position/velocity of mass due to applied forces



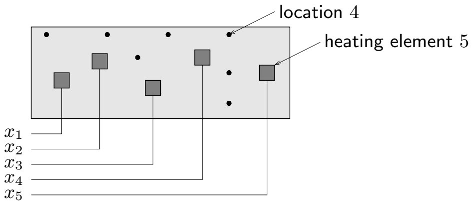
- unit mass, zero position/velocity at $t = 0$, subject to force $f(t)$ for $0 \leq t \leq n$
- $f(t) = x_j$ for $j - 1 \leq t < j$, $j = 1, \ldots, n$
  (x is the sequence of applied forces, constant in each interval)
- y_1, y_2 are final position and velocity ($i.e.$, at $t = n$)
- we have $y = Ax$
- a_{1j} gives influence of applied force during $j - 1 \leq t < j$ on final position
- a_{2j} gives influence of applied force during $j - 1 \leq t < j$ on final velocity

Gravimeter prospecting



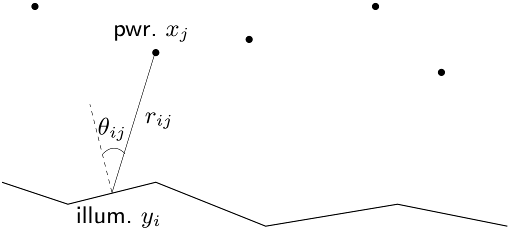
- $x_j = \rho_j - \rho_{\text{avg}}$ is (excess) mass density of earth in voxel j;
- y_i is measured *gravity anomaly* at location i, *i.e.*, some component (typically vertical) of $g_i - g_{\text{avg}}$
- $y = Ax$, where A comes from physics and geometry
- jth column of A shows sensor readings caused by unit density anomaly at voxel j
- ith row of A shows sensitivity pattern of sensor i

Thermal system



- x_j is power of jth heating element or heat source
- y_i is change in steady-state temperature at location i
- thermal transport via conduction
- $y = Ax$
- a_{ij} gives influence of heater j at location i (in $^\circ$C/W)
- jth column of A gives pattern of steady-state temperature rise due to 1W at heater j
- ith row shows how heaters affect location i

Illumination with multiple lamps



- n lamps illuminating m (small, flat) patches, no shadows
- x_j is power of jth lamp; y_i is illumination level of patch i
- $y = Ax$, where $a_{ij} = r_{ij}^{-2} \textbf{max}\{\cos\theta_{ij}, 0\}$
  ($\cos\theta_{ij} < 0$ means patch i is shaded from lamp j)
- jth column of A shows illumination pattern from lamp j

Signal and interference power in wireless system

- n transmitter/receiver pairs
- transmitter j transmits to receiver j (and, inadvertantly, to the other receivers)
- p_j is power of jth transmitter
- s_i is received signal power of ith receiver
- z_i is received interference power of ith receiver
- G_{ij} is path gain from transmitter j to receiver i
- we have $s = Ap$, $z = Bp$, where

$$a_{ij} = \begin{cases} G_{ii} & i = j \\ 0 & i \neq j \end{cases} \qquad b_{ij} = \begin{cases} 0 & i = j \\ G_{ij} & i \neq j \end{cases}$$

- A is diagonal; B has zero diagonal (ideally, A is 'large', B is 'small')

Cost of production

production *inputs* (materials, parts, labor, ...) are combined to make a number of *products*

- x_j is price per unit of production input j
- a_{ij} is units of production input j required to manufacture one unit of product i
- y_i is production cost per unit of product i
- we have $y = Ax$
- ith row of A is *bill of materials* for unit of product i

Cost of production

production inputs needed

- q_i is quantity of product i to be produced
- r_j is total quantity of production input j needed
- we have $r = A^\mathsf{T} q$

total production cost is

$$r^\mathsf{T} x = (A^\mathsf{T} q)^\mathsf{T} x = q^\mathsf{T} A x$$

Network traffic and flows

- n flows with rates $f_1, \ldots, f_n$ pass from their source nodes to their destination nodes over fixed routes in a network
- t_i, traffic on link i, is sum of rates of flows passing through it
- flow routes given by *flow-link incidence matrix*

$$A_{ij} = \begin{cases} 1 & \text{flow } j \text{ goes over link } i \\ 0 & \text{otherwise} \end{cases}$$

- traffic and flow rates related by $t = Af$

Network traffic and flows

link delays and flow latency

- let $d_1, \ldots, d_m$ be link delays, and $l_1, \ldots, l_n$ be latency (total travel time) of flows
- $l = A^\mathsf{T} d$
- $f^\mathsf{T} l = f^\mathsf{T} A^\mathsf{T} d = (Af)^\mathsf{T} d = t^\mathsf{T} d$, total # of packets in network

Linearization

- if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x_0 \in \mathbb{R}^n$, then

$$x \text{ near } x_0 \Longrightarrow f(x) \text{ very near } f(x_0) + Df(x_0)(x - x_0)$$

  where

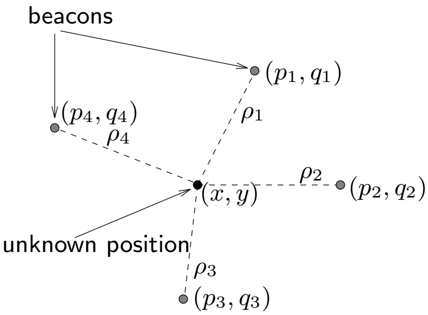
$$Df(x_0)_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{x_0}$$

  is derivative (Jacobian) matrix

- with $y = f(x)$, $y_0 = f(x_0)$, define *input deviation* $\delta x := x - x_0$, *output deviation* $\delta y := y - y_0$
- then we have $\delta y \approx Df(x_0)\delta x$
- when deviations are small, they are (approximately) related by a linear function

Navigation by range measurement

- (x, y) unknown coordinates in plane
- (p_i, q_i) known coordinates of beacons for $i = 1, 2, 3, 4$
- ρ_i measured (known) distance or range from beacon i



Navigation by range measurement

- $\rho \in \mathbb{R}^4$ is a nonlinear function of $(x, y) \in \mathbb{R}^2$

$$\rho_i(x, y) = \sqrt{(x - p_i)^2 + (y - q_i)^2}$$

- linearize around (x_0, y_0): $\delta\rho \approx A \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$, where

$$a_{i1} = \frac{(x_0 - p_i)}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}, \quad a_{i2} = \frac{(y_0 - q_i)}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}$$

- ith row of A shows (approximate) change in ith range measurement for (small) shift in (x, y) from (x_0, y_0)
- first column of A shows sensitivity of range measurements to (small) change in x from x_0
- obvious application: (x_0, y_0) is last navigation fix; (x, y) is current position, a short time later