

# Engineering examples

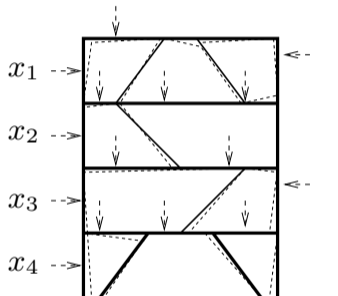
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## Linear elastic structure

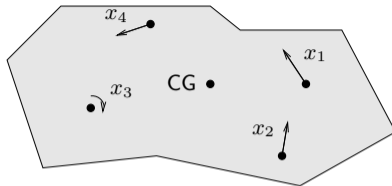
- ▶  $x_j$  is external force applied at some node, in some fixed direction
- ▶  $y_i$  is (small) deflection of some node, in some fixed direction



(provided  $x, y$  are small) we have  $y \approx Ax$

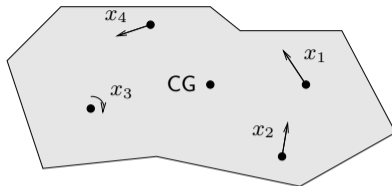
- ▶  $A$  is called the *compliance matrix*
- ▶  $a_{ij}$  gives deflection  $i$  per unit force at  $j$  (in m/N)

## Total force/torque on rigid body



- ▶  $x_j$  is external force/torque applied at some point/direction/axis
- ▶  $y \in \mathbb{R}^6$  is resulting total force & torque on body  
( $y_1, y_2, y_3$  are x-, y-, z- components of total force,  
 $y_4, y_5, y_6$  are x-, y-, z- components of total torque)
- ▶ we have  $y = Ax$
- ▶  $A$  depends on geometry  
(of applied forces and torques with respect to center of gravity CG)
- ▶  $j$ th column gives resulting force & torque for unit force/torque  $j$

## Total force/torque on rigid body



► force  $x$  applied in direction  $d$  at point  $r$  relative to CG

► resulting force is

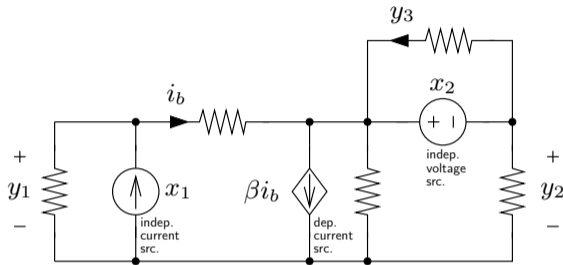
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

► resulting torque is

$$\begin{bmatrix} y_4 \\ y_5 \\ y_6 \end{bmatrix} = x(r \times d) = x \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

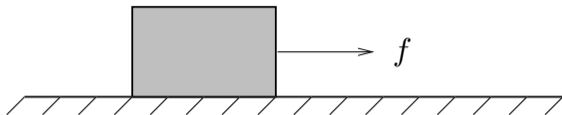
## Linear static circuit

interconnection of resistors, linear dependent (controlled) sources, and independent sources



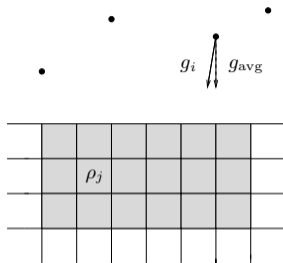
- ▶  $x_j$  is value of independent source  $j$
- ▶  $y_i$  is some circuit variable (voltage, current)
- ▶ we have  $y = Ax$
- ▶ if  $x_j$  are currents and  $y_i$  are voltages,  $A$  is called the *impedance* or *resistance* matrix

## Final position/velocity of mass due to applied forces



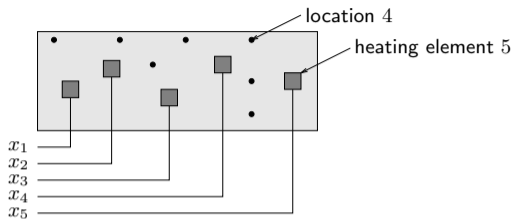
- ▶ unit mass, zero position/velocity at  $t = 0$ , subject to force  $f(t)$  for  $0 \leq t \leq n$
- ▶  $f(t) = x_j$  for  $j - 1 \leq t < j$ ,  $j = 1, \dots, n$   
( $x$  is the sequence of applied forces, constant in each interval)
- ▶  $y_1, y_2$  are final position and velocity (*i.e.*, at  $t = n$ )
- ▶ we have  $y = Ax$
- ▶  $a_{1j}$  gives influence of applied force during  $j - 1 \leq t < j$  on final position
- ▶  $a_{2j}$  gives influence of applied force during  $j - 1 \leq t < j$  on final velocity

## Gravimeter prospecting



- ▶  $x_j = \rho_j - \rho_{avg}$  is (excess) mass density of earth in voxel  $j$ ;
- ▶  $y_i$  is measured *gravity anomaly* at location  $i$ , *i.e.*, some component (typically vertical) of  $g_i - g_{avg}$
- ▶  $y = Ax$ , where  $A$  comes from physics and geometry
- ▶  $j$ th column of  $A$  shows sensor readings caused by unit density anomaly at voxel  $j$
- ▶  $i$ th row of  $A$  shows sensitivity pattern of sensor  $i$

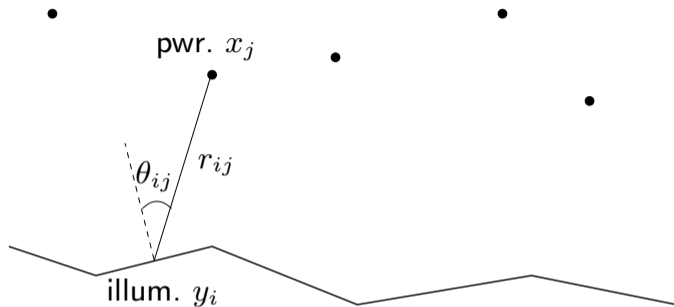
## Thermal system



- ▶  $x_j$  is power of  $j$ th heating element or heat source
- ▶  $y_i$  is change in steady-state temperature at location  $i$
- ▶ thermal transport via conduction
- ▶  $y = Ax$
- ▶  $a_{ij}$  gives influence of heater  $j$  at location  $i$  (in  $^{\circ}\text{C}/\text{W}$ )
- ▶  $j$ th column of  $A$  gives pattern of steady-state temperature rise due to 1W at heater  $j$
- ▶  $i$ th row shows how heaters affect location  $i$



## Illumination with multiple lamps



- ▶  $n$  lamps illuminating  $m$  (small, flat) patches, no shadows
- ▶  $x_j$  is power of  $j$ th lamp;  $y_i$  is illumination level of patch  $i$
- ▶  $y = Ax$ , where  $a_{ij} = r_{ij}^{-2} \max\{\cos\theta_{ij}, 0\}$   
( $\cos\theta_{ij} < 0$  means patch  $i$  is shaded from lamp  $j$ )
- ▶  $j$ th column of  $A$  shows illumination pattern from lamp  $j$

## Signal and interference power in wireless system

- ▶  $n$  transmitter/receiver pairs
- ▶ transmitter  $j$  transmits to receiver  $j$  (and, inadvertently, to the other receivers)
- ▶  $p_j$  is power of  $j$ th transmitter
- ▶  $s_i$  is received signal power of  $i$ th receiver
- ▶  $z_i$  is received interference power of  $i$ th receiver
- ▶  $G_{ij}$  is path gain from transmitter  $j$  to receiver  $i$
- ▶ we have  $s = Ap$ ,  $z = Bp$ , where

$$a_{ij} = \begin{cases} G_{ii} & i = j \\ 0 & i \neq j \end{cases} \quad b_{ij} = \begin{cases} 0 & i = j \\ G_{ij} & i \neq j \end{cases}$$

- ▶  $A$  is diagonal;  $B$  has zero diagonal (ideally,  $A$  is 'large',  $B$  is 'small')

## Cost of production

production *inputs* (materials, parts, labor, ...) are combined to make a number of *products*

- ▶  $x_j$  is price per unit of production input  $j$
- ▶  $a_{ij}$  is units of production input  $j$  required to manufacture one unit of product  $i$
- ▶  $y_i$  is production cost per unit of product  $i$
- ▶ we have  $y = Ax$
- ▶  $i$ th row of  $A$  is *bill of materials* for unit of product  $i$

## Cost of production

### production inputs needed

- ▶  $q_i$  is quantity of product  $i$  to be produced
- ▶  $r_j$  is total quantity of production input  $j$  needed
- ▶ we have  $r = A^T q$

total production cost is

$$r^T x = (A^T q)^T x = q^T Ax$$

$A$  maps prices of production inputs to prices of products,  $A^T$  maps quantities of products to quantities of production inputs

## Network traffic and flows

- ▶  $n$  flows with rates  $f_1, \dots, f_n$  pass from their source nodes to their destination nodes over fixed routes in a network
- ▶  $t_i$ , traffic on link  $i$ , is sum of rates of flows passing through it
- ▶ flow routes given by *flow-link incidence matrix*

$$A_{ij} = \begin{cases} 1 & \text{flow } j \text{ goes over link } i \\ 0 & \text{otherwise} \end{cases}$$

- ▶ traffic and flow rates related by  $t = Af$

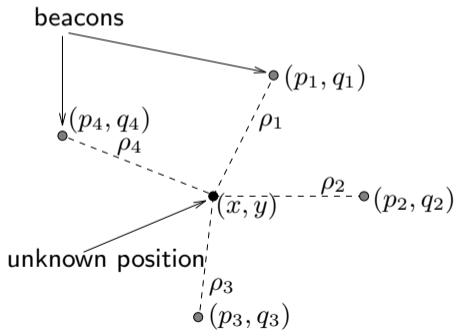
## Network traffic and flows

### link delays and flow latency

- ▶ let  $d_1, \dots, d_m$  be link delays, and  $l_1, \dots, l_n$  be latency (total travel time) of flows
- ▶  $l = A^T d$
- ▶  $f^T l = f^T A^T d = (Af)^T d = t^T d$ , total # of packets in network

## Navigation by range measurement

- ▶  $(x, y)$  unknown coordinates in plane
- ▶  $(p_i, q_i)$  known coordinates of beacons for  $i = 1, 2, 3, 4$
- ▶  $\rho_i$  measured (known) distance or range from beacon  $i$



## Navigation by range measurement

- ▶  $\rho \in \mathbb{R}^4$  is a nonlinear function of  $(x, y) \in \mathbb{R}^2$

$$\rho_i(x, y) = \sqrt{(x - p_i)^2 + (y - q_i)^2}$$

- ▶ linearize around  $(x_0, y_0)$ :  $\delta\rho \approx A \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$ , where

$$a_{i1} = \frac{(x_0 - p_i)}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}, \quad a_{i2} = \frac{(y_0 - q_i)}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}$$

- ▶  $i$ th row of  $A$  shows (approximate) change in  $i$ th range measurement for (small) shift in  $(x, y)$  from  $(x_0, y_0)$
- ▶ first column of  $A$  shows sensitivity of range measurements to (small) change in  $x$  from  $x_0$
- ▶ obvious application:  $(x_0, y_0)$  is last navigation fix;  $(x, y)$  is current position, a short time later