Engineering examples
Linear elastic structure

- $x_j$ is external force applied at some node, in some fixed direction
- $y_i$ is (small) deflection of some node, in some fixed direction

(provided $x$, $y$ are small) we have $y \approx Ax$

- $A$ is called the **compliance matrix**
- $a_{ij}$ gives deflection $i$ per unit force at $j$ (in m/N)
Total force/torque on rigid body

- $x_j$ is external force/torque applied at some point/direction/axis
- $y \in \mathbb{R}^6$ is resulting total force & torque on body
  ($y_1, y_2, y_3$ are $x$-, $y$-, $z$- components of total force,
  $y_4, y_5, y_6$ are $x$-, $y$-, $z$- components of total torque)
- we have $y = Ax$
- $A$ depends on geometry
  (of applied forces and torques with respect to center of gravity CG)
- $j$th column gives resulting force & torque for unit force/torque $j$
Linear static circuit

interconnection of resistors, linear dependent (controlled) sources, and independent sources

$\mathbf{x}_j$ is value of independent source $j$

$y_i$ is some circuit variable (voltage, current)

we have $y = Ax$

if $x_j$ are currents and $y_i$ are voltages, $A$ is called the \textit{impedance} or \textit{resistance} matrix
Final position/velocity of mass due to applied forces

- unit mass, zero position/velocity at $t = 0$, subject to force $f(t)$ for $0 \leq t \leq n$

- $f(t) = x_j$ for $j - 1 \leq t < j$, $j = 1, \ldots, n$
  ($x$ is the sequence of applied forces, constant in each interval)

- $y_1, y_2$ are final position and velocity (i.e., at $t = n$)

- we have $y = Ax$

- $a_{1j}$ gives influence of applied force during $j - 1 \leq t < j$ on final position

- $a_{2j}$ gives influence of applied force during $j - 1 \leq t < j$ on final velocity
Gravimeter prospecting

\[ x_j = \rho_j - \rho_{avg} \] is (excess) mass density of earth in voxel \( j \);

\( y_i \) is measured gravity anomaly at location \( i \), i.e., some component (typically vertical) of \( g_i - g_{avg} \)

\( y = Ax \), where \( A \) comes from physics and geometry

\( j \)th column of \( A \) shows sensor readings caused by unit density anomaly at voxel \( j \)

\( i \)th row of \( A \) shows sensitivity pattern of sensor \( i \)
Thermal system

- $x_j$ is power of $j$th heating element or heat source
- $y_i$ is change in steady-state temperature at location $i$
- thermal transport via conduction
- $y = Ax$
- $a_{ij}$ gives influence of heater $j$ at location $i$ (in °C/W)
- $j$th column of $A$ gives pattern of steady-state temperature rise due to 1W at heater $j$
- $i$th row shows how heaters affect location $i$
Illumination with multiple lamps

- $n$ lamps illuminating $m$ (small, flat) patches, no shadows
- $x_j$ is power of $j$th lamp; $y_i$ is illumination level of patch $i$
- $y = Ax$, where $a_{ij} = r_{ij}^{-2} \max\{\cos \theta_{ij}, 0\}$
  - $(\cos \theta_{ij} < 0$ means patch $i$ is shaded from lamp $j$)
- $j$th column of $A$ shows illumination pattern from lamp $j$
Signal and interference power in wireless system

- \( n \) transmitter/receiver pairs
- Transmitter \( j \) transmits to receiver \( j \) (and, inadvertently, to the other receivers)
- \( p_j \) is power of \( j \)th transmitter
- \( s_i \) is received signal power of \( i \)th receiver
- \( z_i \) is received interference power of \( i \)th receiver
- \( G_{ij} \) is path gain from transmitter \( j \) to receiver \( i \)
- We have \( s = Ap, z = Bp \), where
  \[
  a_{ij} = \begin{cases} 
  G_{ii} & i = j \\ 
  0 & i \neq j 
  \end{cases} \quad b_{ij} = \begin{cases} 
  0 & i = j \\ 
  G_{ij} & i \neq j 
  \end{cases}
  \]
- \( A \) is diagonal; \( B \) has zero diagonal (ideally, \( A \) is ‘large’, \( B \) is ‘small’).
Cost of production

production inputs (materials, parts, labor, ...) are combined to make a number of products

- $x_j$ is price per unit of production input $j$
- $a_{ij}$ is units of production input $j$ required to manufacture one unit of product $i$
- $y_i$ is production cost per unit of product $i$
- We have $y = Ax$
- $i$th row of $A$ is bill of materials for unit of product $i$
Cost of production

Production inputs needed

- $q_i$ is quantity of product $i$ to be produced
- $r_j$ is total quantity of production input $j$ needed
- we have $r = A^T q$

Total production cost is

$$r^T x = (A^T q)^T x = q^T A x$$
Network traffic and flows

- $n$ flows with rates $f_1, \ldots, f_n$ pass from their source nodes to their destination nodes over fixed routes in a network.

- $t_i$, traffic on link $i$, is sum of rates of flows passing through it.

- Flow routes given by flow-link incidence matrix $A_{ij}$:

  $A_{ij} = \begin{cases} 
  1 & \text{flow } j \text{ goes over link } i \\
  0 & \text{otherwise}
  \end{cases}$

- Traffic and flow rates related by $t = Af$. 

Network traffic and flows

link delays and flow latency

- let $d_1, \ldots, d_m$ be link delays, and $l_1, \ldots, l_n$ be latency (total travel time) of flows
- $l = A^T d$
- $f^T l = f^T A^T d = (Af)^T d = t^T d$, total # of packets in network
Navigation by range measurement

- $(x, y)$ unknown coordinates in plane
- $(p_i, q_i)$ known coordinates of beacons for $i = 1, 2, 3, 4$
- $\rho_i$ measured (known) distance or range from beacon $i$
Navigation by range measurement

- $\rho \in \mathbb{R}^4$ is a nonlinear function of $(x, y) \in \mathbb{R}^2$

$$\rho_i(x, y) = \sqrt{(x - p_i)^2 + (y - q_i)^2}$$

- Linearize around $(x_0, y_0)$: $\delta \rho \approx A \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$, where

$$a_{i1} = \frac{(x_0 - p_i)}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}, \quad a_{i2} = \frac{(y_0 - q_i)}{\sqrt{(x_0 - p_i)^2 + (y_0 - q_i)^2}}$$

- $i$th row of $A$ shows (approximate) change in $i$th range measurement for (small) shift in $(x, y)$ from $(x_0, y_0)$

- First column of $A$ shows sensitivity of range measurements to (small) change in $x$ from $x_0$

- Obvious application: $(x_0, y_0)$ is last navigation fix; $(x, y)$ is current position, a short time later