Ellipsoids

- ellipsoids
- ellipsoids in estimation
Ellipsoids

if $A = A^T > 0$, the set

$$\mathcal{E} = \{ x \mid x^T A x \leq 1 \}$$

is an ellipsoid in $\mathbb{R}^n$, centered at 0.
Ellipsoids

semi-axes are given by $s_i = \lambda_i^{-1/2} q_i$, i.e.:

- eigenvectors determine directions of semiaxes
- eigenvalues determine lengths of semiaxes

note:

- in direction $q_1$, $x^T Ax$ is *large*, hence ellipsoid is *thin* in direction $q_1$
- in direction $q_n$, $x^T Ax$ is *small*, hence ellipsoid is *fat* in direction $q_n$
- $\sqrt{\lambda_{\text{max}}/\lambda_{\text{min}}}$ gives maximum *eccentricity*

if $\mathcal{E} = \{ x \mid x^T B x \leq 1 \}$, where $B > 0$, then $\mathcal{E} \subseteq \tilde{\mathcal{E}} \iff A \succeq B$
Ellipsoids in estimation

- if \( y = Ax \) then \( \|y\|^2 = x^T A^T Ax \) so
  \[
  \{ x \in \mathbb{R}^n \mid x^T A^T Ax \leq 1 \} \quad \text{maps into} \quad \{ y \in \mathbb{R}^m \mid \|y\| \leq 1 \}
  \]

- assume \( A \) is skinny and full rank, then \( A^T A \) is positive definite

- \( \sigma_i^2 = i \text{th eigenvalue of } A^T A \); convention \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0 \), called singular values of \( A \)

- \( v_i \) is unit eigenvector of \( A^T A \) corresponding to \( \sigma_i \)

- \( u_i = \frac{1}{\sigma_i} A v_i \), are orthogonal
Ellipsoids in estimation

- Short axis of ellipsoid (eigenvector $v_1$ corresponding $\lambda_{\text{max}}(A^T A)$) is stretched most by sensing.
- Long axis of ellipsoid (eigenvector $v_n$ corresponding $\lambda_{\text{min}}(A^T A)$) is stretched least by sensing.

Therefore

- Small changes to $x$ in the direction $v_1$ cause large changes in sensor readings $y$; sensors are highly sensitive
- Small changes to $x$ in the direction $v_n$ cause small changes in sensor readings $y$; sensors are insensitive
Example: Navigation

Here $A \in \mathbb{R}^{4 \times 2}$ with

$$A = \begin{bmatrix} b_1^T \\ b_2^T \\ b_3^T \\ b_4^T \end{bmatrix}$$

and $y = Ax$. Each $b_i$ is a unit vector.

- $x$ is unknown.
- $y$ is measured, with $y_i$ the component of $x$ in the direction $b_i$.
- the ellipsoid is the set of $x \in \mathbb{R}^2$ which result in $\|y\| \leq 1$
- plot shows contours of $\|y\|$, i.e., contours of $\sqrt{x^T A^T A x}$
Example: Row interpretation

\[ A = \begin{bmatrix} b_1^T \\ b_2^T \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0.25 & 1 \end{bmatrix} \]

\( v_1 \) and \( v_2 \) are eigenvectors of \( A^T A \) with corresponding singular values \( \sigma_1 \approx 1.5117 \) and \( \sigma_2 \approx 0.1654 \)

sensors are approx 10 times more sensitive to changes in \( x \) in the \( v_1 \) direction than in \( v_2 \) direction.