

Ellipsoids

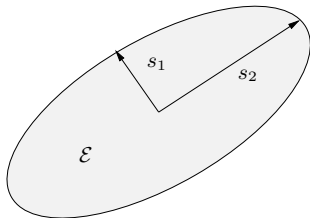
- ▶ ellipsoids
- ▶ ellipsoids in estimation

Ellipsoids

if $A = A^T > 0$, the set

$$\mathcal{E} = \{ x \mid x^T A x \leq 1 \}$$

is an *ellipsoid* in \mathbb{R}^n , centered at 0



Ellipsoids

semi-axes are given by $s_i = \lambda_i^{-1/2} q_i$, i.e.:

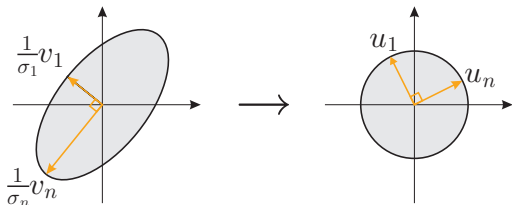
- ▶ eigenvectors determine directions of semiaxes
- ▶ eigenvalues determine lengths of semiaxes

note:

- ▶ in direction q_1 , $x^T Ax$ is *large*, hence ellipsoid is *thin* in direction q_1
- ▶ in direction q_n , $x^T Ax$ is *small*, hence ellipsoid is *fat* in direction q_n
- ▶ $\sqrt{\lambda_{\max}/\lambda_{\min}}$ gives maximum *eccentricity*

if $\tilde{\mathcal{E}} = \{ x \mid x^T B x \leq 1 \}$, where $B > 0$, then $\mathcal{E} \subseteq \tilde{\mathcal{E}} \iff A \geq B$

Ellipsoids in estimation

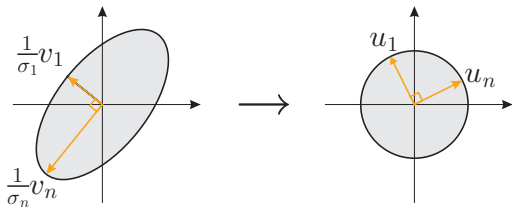


- ▶ if $y = Ax$ then $\|y\|^2 = x^T A^T A x$ so

$$\{x \in \mathbb{R}^n \mid x^T A^T A x \leq 1\} \quad \text{maps into} \quad \{y \in \mathbb{R}^m \mid \|y\| \leq 1\}$$

- ▶ assume A is skinny and full rank, then $A^T A$ is positive definite
- ▶ $\sigma_i^2 = i$ th eigenvalue of $A^T A$; convention $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$, called *singular values* of A
- ▶ v_i is unit eigenvector of $A^T A$ corresponding to σ_i
- ▶ $u_i = \frac{1}{\sigma_i} A v_i$, are orthogonal

Ellipsoids in estimation



- ▶ short axis of ellipsoid (eigenvector v_1 corresponding $\lambda_{\max}(A^T A)$) is *stretched most* by sensing.
- ▶ long axis of ellipsoid (eigenvector v_n corresponding $\lambda_{\min}(A^T A)$) is *stretched least* by sensing.

therefore

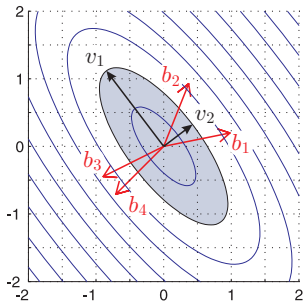
- ▶ small changes to x in the direction v_1 cause large changes in sensor readings y ; sensors are *highly sensitive*
- ▶ small changes to x in the direction v_n cause small changes in sensor readings y ; sensors are *insensitive*

Example: Navigation

here $A \in \mathbb{R}^{4 \times 2}$ with

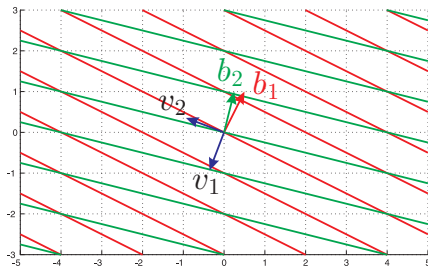
$$A = \begin{bmatrix} b_1^T \\ b_2^T \\ b_3^T \\ b_4^T \end{bmatrix}$$

and $y = Ax$. Each b_i is a unit vector.



- ▶ x is unknown.
- ▶ y is measured, with y_i the component of x in the direction b_i
- ▶ the ellipsoid is the set of $x \in \mathbb{R}^2$ which result in $\|y\| \leq 1$
- ▶ plot shows contours of $\|y\|$, i.e., contours of $\sqrt{x^T A^T A x}$

Example: Row interpretation



$$A = \begin{bmatrix} b_1^T \\ b_2^T \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0.25 & 1 \end{bmatrix}$$

v_1 and v_2 are eigenvectors of $A^T A$ with corresponding singular values

$$\sigma_1 \approx 1.5117 \quad \sigma_2 \approx 0.1654$$

sensors are approx 10 times more sensitive to changes in x in the v_1 direction than in v_2 direction.