Ellipsoids

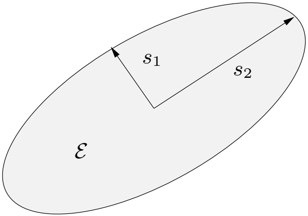
- ellipsoids
- ellipsoids in estimation

Ellipsoids

if $A = A^\mathsf{T} > 0$, the set

$$\mathcal{E} = \{ \ x \mid x^\mathsf{T} A x \leq 1 \ \}$$

is an *ellipsoid* in $\mathbb{R}^n$, centered at 0



Ellipsoids

semi-axes are given by $s_i = \lambda_i^{-1/2} q_i$, *i.e.*:

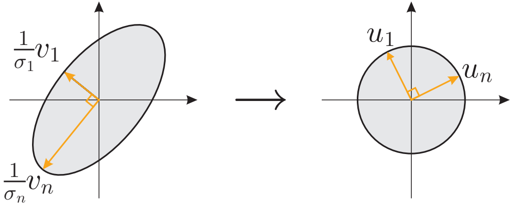
- eigenvectors determine directions of semiaxes
- eigenvalues determine lengths of semiaxes

note:

- in direction q_1, $x^\mathsf{T} A x$ is *large*, hence ellipsoid is *thin* in direction q_1
- in direction q_n, $x^\mathsf{T} A x$ is *small*, hence ellipsoid is *fat* in direction q_n
- $\sqrt{\lambda_{\text{max}}/\lambda_{\text{min}}}$ gives maximum *eccentricity*

if $\tilde{\mathcal{E}} = \{ \, x \mid x^\mathsf{T} B x \leq 1 \, \}$, where $B > 0$, then $\mathcal{E} \subseteq \tilde{\mathcal{E}} \iff A \geq B$

Ellipsoids in estimation

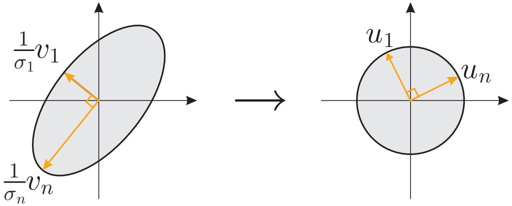


- if $y = Ax$ then $\|y\|^2 = x^T A^T A x$ so

$$\{ x \in \mathbb{R}^n \mid x^T A^T A x \leq 1 \} \quad \text{maps into} \quad \{ y \in \mathbb{R}^m \mid \|y\| \leq 1 \}$$

- assume A is skinny and full rank, then $A^T A$ is positive definite
- $\sigma_i^2 = i$th eigenvalue of $A^T A$; convention $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$, called *singular values* of A
- v_i is unit eigenvector of $A^T A$ corresponding to σ_i
- $u_i = \frac{1}{\sigma_i} A v_i$, are orthogonal

Ellipsoids in estimation



- short axis of ellipsoid (eigenvector v_1 corresponding $\lambda_{\text{max}}(A^T A)$) is *stretched most* by sensing.
- long axis of ellipsoid (eigenvector v_n corresponding $\lambda_{\text{min}}(A^T A)$) is *stretched least* by sensing.

therefore

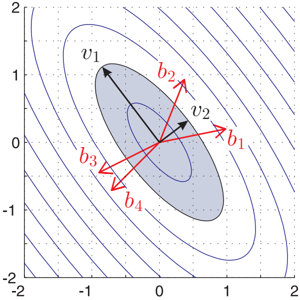
- small changes to x in the direction v_1 cause large changes in sensor readings y; sensors are *highly sensitive*
- small changes to x in the direction v_n cause small changes in sensor readings y; sensors are *insensitive*

Example: Navigation

here $A \in \mathbb{R}^{4\times 2}$ with

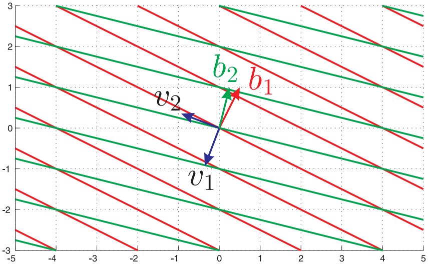
$$A = \begin{bmatrix} b_1^T \\ b_2^T \\ b_3^T \\ b_4^T \end{bmatrix}$$

and $y = Ax$. Each b_i is a unit vector.



- x is unknown.
- y is measured, with y_i the component of x in the direction b_i
- the ellipsoid is the set of $x \in \mathbb{R}^2$ which result in $\|y\| \le 1$
- plot shows contours of $\|y\|$, i.e., contours of $\sqrt{x^T A^T A x}$

Example: Row interpretation



$$A = \begin{bmatrix} b_1^T \\ b_2^T \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0.25 & 1 \end{bmatrix}$$

v_1 and v_2 are eigenvectors of A^TA with corresponding singular values

$$\sigma_1 \approx 1.5117 \qquad \sigma_2 \approx 0.1654$$

sensors are approx 10 times more sensitive to changes in x in the v_1 direction than in v_2 direction.