

# Eigenvectors and diagonalization

- ▶ eigenvectors
- ▶ diagonalization

## Eigenvectors and eigenvalues

$\lambda \in \mathbb{C}$  is called an *eigenvalue* of  $A \in \mathbb{C}^{n \times n}$  if

$$\mathcal{X}(\lambda) = \det(\lambda I - A) = 0$$

equivalent to:

- ▶ there exists nonzero  $v \in \mathbb{C}^n$  s.t.  $(\lambda I - A)v = 0$ , *i.e.*,

$$Av = \lambda v$$

any such  $v$  is called an *eigenvector* of  $A$  (associated with eigenvalue  $\lambda$ )

- ▶ there exists nonzero  $w \in \mathbb{C}^n$  s.t.  $w^T(\lambda I - A) = 0$ , *i.e.*,

$$w^T A = \lambda w^T$$

any such  $w$  is called a *left eigenvector* of  $A$

## Complex eigenvalues and eigenvectors

- ▶ if  $v$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ , then so is  $\alpha v$ , for any  $\alpha \in \mathbb{C}$ ,  $\alpha \neq 0$
- ▶ even when  $A$  is real, eigenvalue  $\lambda$  and eigenvector  $v$  can be complex
- ▶ when  $A$  and  $\lambda$  are real, we can always find a real eigenvector  $v$  associated with  $\lambda$ : if  $Av = \lambda v$ , with  $A \in \mathbb{R}^{n \times n}$ ,  $\lambda \in \mathbb{R}$ , and  $v \in \mathbb{C}^n$ , then

$$A\Re v = \lambda\Re v, \quad A\Im v = \lambda\Im v$$

so  $\Re v$  and  $\Im v$  are real eigenvectors, if they are nonzero (and at least one is)

- ▶ *conjugate symmetry*: if  $A$  is real and  $v \in \mathbb{C}^n$  is an eigenvector associated with  $\lambda \in \mathbb{C}$ , then  $\bar{v}$  is an eigenvector associated with  $\bar{\lambda}$ :  
taking conjugate of  $Av = \lambda v$  we get  $\overline{Av} = \overline{\lambda v}$ , so

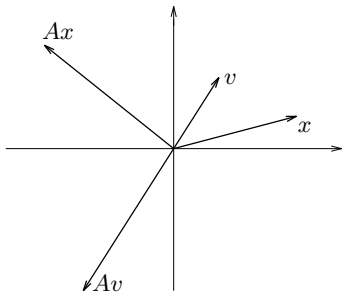
$$A\bar{v} = \bar{\lambda}\bar{v}$$

we'll assume  $A$  is real from now on ...

## Scaling interpretation

(assume  $\lambda \in \mathbb{R}$  for now; we'll consider  $\lambda \in \mathbb{C}$  later)

if  $v$  is an eigenvector, effect of  $A$  on  $v$  is very simple: scaling by  $\lambda$



## Scaling

- ▶  $\lambda \in \mathbb{R}, \lambda > 0$ :  $v$  and  $Av$  point in same direction
- ▶  $\lambda \in \mathbb{R}, \lambda < 0$ :  $v$  and  $Av$  point in opposite directions
- ▶  $\lambda \in \mathbb{R}, |\lambda| < 1$ :  $Av$  smaller than  $v$
- ▶  $\lambda \in \mathbb{R}, |\lambda| > 1$ :  $Av$  larger than  $v$

(we'll see later how this relates to stability of continuous- and discrete-time systems...)

## Diagonalization

suppose  $v_1, \dots, v_n$  is a *linearly independent* set of eigenvectors of  $A \in \mathbb{R}^{n \times n}$ :

$$Av_i = \lambda_i v_i, \quad i = 1, \dots, n$$

express as

$$A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

define  $T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$  and  $\Lambda = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$ , so

$$AT = T\Lambda$$

## Diagonalization

$$T^{-1}AT = \Lambda$$

- ▶  $T$  invertible means  $v_1, \dots, v_n$  linearly independent
- ▶ similarity transformation by  $T$  *diagonalizes*  $A$
- ▶ existence of invertible  $T$  such that

$$T^{-1}AT = \Lambda = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$$

is equivalent to existence of a linearly independent set of  $n$  eigenvectors

- ▶ we say  $A$  is *diagonalizable*
- ▶ if  $A$  is not diagonalizable, it is sometimes called *defective*

## Not all matrices are diagonalizable

**example:**  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

- ▶ characteristic polynomial is  $\mathcal{X}(s) = s^2$ , so  $\lambda = 0$  is only eigenvalue
- ▶ eigenvectors satisfy  $Av = 0v = 0$ , *i.e.*

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

- ▶ so all eigenvectors have form  $v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$  where  $v_1 \neq 0$
- ▶ thus,  $A$  cannot have two independent eigenvectors



## Distinct eigenvalues

if  $A$  has distinct eigenvalues then  $A$  is diagonalizable

- ▶ distinct eigenvalues means  $\lambda_i \neq \lambda_j$  for  $i \neq j$
- ▶ the converse is false —  $A$  can have repeated eigenvalues but still be diagonalizable

## Diagonalization and left eigenvectors

rewrite  $T^{-1}AT = \Lambda$  as  $T^{-1}A = \Lambda T^{-1}$ , or

$$\begin{bmatrix} w_1^\top \\ \vdots \\ w_n^\top \end{bmatrix} A = \Lambda \begin{bmatrix} w_1^\top \\ \vdots \\ w_n^\top \end{bmatrix}$$

where  $w_1^\top, \dots, w_n^\top$  are the rows of  $T^{-1}$

thus

$$w_i^\top A = \lambda_i w_i^\top$$

*i.e.*, the rows of  $T^{-1}$  are (lin. indep.) left eigenvectors, normalized so that

$$w_i^\top v_j = \delta_{ij}$$

(*i.e.*, left & right eigenvectors chosen this way are *dual bases*)