# Eigenvectors and diagonalization

- eigenvectors
- diagonalization

## Eigenvectors and eigenvalues

$\lambda \in \mathbb{C}$ is called an *eigenvalue* of $A \in \mathbb{C}^{n \times n}$ if

$$\mathcal{X}(\lambda) = \det(\lambda I - A) = 0$$

equivalent to:

- there exists nonzero $v \in \mathbb{C}^n$ s.t. $(\lambda I - A)v = 0$, *i.e.*,

  $$Av = \lambda v$$

  any such $v$ is called an *eigenvector* of $A$ (associated with eigenvalue $\lambda$)

- there exists nonzero $w \in \mathbb{C}^n$ s.t. $w^T(\lambda I - A) = 0$, *i.e.*,

  $$w^T A = \lambda w^T$$

  any such $w$ is called a *left eigenvector* of $A$

## Complex eigenvalues and eigenvectors

- if $v$ is an eigenvector of $A$ with eigenvalue $\lambda$, then so is $\alpha v$, for any $\alpha \in \mathbb{C}$, $\alpha \neq 0$
- even when $A$ is real, eigenvalue $\lambda$ and eigenvector $v$ can be complex
- when $A$ and $\lambda$ are real, we can always find a real eigenvector $v$ associated with $\lambda$: if $Av = \lambda v$, with $A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{R}$, and $v \in \mathbb{C}^n$, then

  $$A\Re v = \lambda \Re v, \qquad A\Im v = \lambda \Im v$$

  so $\Re v$ and $\Im v$ are real eigenvectors, if they are nonzero
  (and at least one is)
- *conjugate symmetry*: if $A$ is real and $v \in \mathbb{C}^n$ is an eigenvector associated with $\lambda \in \mathbb{C}$, then $\overline{v}$ is an eigenvector associated with $\overline{\lambda}$:

  taking conjugate of $Av = \lambda v$ we get $\overline{Av} = \overline{\lambda v}$, so

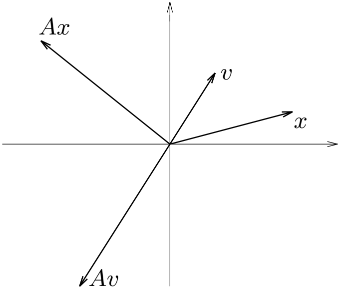
  $$A\overline{v} = \overline{\lambda}\,\overline{v}$$

**we'll assume $A$ is real from now on . . .**

## Scaling interpretation

(assume $\lambda \in \mathbb{R}$ for now; we'll consider $\lambda \in \mathbb{C}$ later)

if $v$ is an eigenvector, effect of $A$ on $v$ is very simple: scaling by $\lambda$



## Scaling

- $\lambda \in \mathbb{R}$, $\lambda > 0$: $v$ and $Av$ point in same direction
- $\lambda \in \mathbb{R}$, $\lambda < 0$: $v$ and $Av$ point in opposite directions
- $\lambda \in \mathbb{R}$, $|\lambda| < 1$: $Av$ smaller than $v$
- $\lambda \in \mathbb{R}$, $|\lambda| > 1$: $Av$ larger than $v$

(we'll see later how this relates to stability of continuous- and discrete-time systems...)

## Diagonalization

suppose $v_1, \ldots, v_n$ is a *linearly independent* set of eigenvectors of $A \in \mathbb{R}^{n \times n}$:

$$Av_i = \lambda_i v_i, \quad i = 1, \ldots, n$$

express as

$$A \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right] = \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right] \left[\begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{array}\right]$$

define $T = \left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array}\right]$ and $\Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$, so

$$\boxed{AT = T\Lambda}$$

## Diagonalization

$$\boxed{T^{-1}AT = \Lambda}$$

- $T$ invertible means $v_1, \ldots, v_n$ linearly independent
- similarity transformation by $T$ *diagonalizes* $A$
- existence of invertible $T$ such that

$$T^{-1}AT = \Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$$

  is equivalent to existence of a linearly independent set of $n$ eigenvectors
- we say $A$ is *diagonalizable*
- if $A$ is not diagonalizable, it is sometimes called *defective*

# Not all matrices are diagonalizable

**example:** $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

- characteristic polynomial is $\mathcal{X}(s) = s^2$, so $\lambda = 0$ is only eigenvalue
- eigenvectors satisfy $Av = 0v = 0$, *i.e.*

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

- so all eigenvectors have form $v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$ where $v_1 \neq 0$
- thus, $A$ cannot have two independent eigenvectors

## Distinct eigenvalues

if $A$ has distinct eigenvalues then $A$ is diagonalizable

- distinct eigenvalues means $\lambda_i \neq \lambda_j$ for $i \neq j$
- the converse is false — $A$ can have repeated eigenvalues but still be diagonalizable

## Diagonalization and left eigenvectors

rewrite $T^{-1}AT = \Lambda$ as $T^{-1}A = \Lambda T^{-1}$, or

$$\begin{bmatrix} w_1^\mathsf{T} \\ \vdots \\ w_n^\mathsf{T} \end{bmatrix} A = \Lambda \begin{bmatrix} w_1^\mathsf{T} \\ \vdots \\ w_n^\mathsf{T} \end{bmatrix}$$

where $w_1^\mathsf{T}, \ldots, w_n^\mathsf{T}$ are the rows of $T^{-1}$

thus

$$w_i^\mathsf{T} A = \lambda_i w_i^\mathsf{T}$$

$i.e.$, the rows of $T^{-1}$ are (lin. indep.) left eigenvectors, normalized so that

$$w_i^\mathsf{T} v_j = \delta_{ij}$$

($i.e.$, left & right eigenvectors chosen this way are *dual bases*)