Dynamic interpretation of eigenvectors

- invariant sets
- complex eigenvectors & invariant planes
- left eigenvectors
- modal form
- discrete-time stability
Dynamic interpretation

suppose $Av = \lambda v$, $v \neq 0$

if $\dot{x} = Ax$ and $x(0) = v$, then $x(t) = e^{\lambda t}v$

several ways to see this, e.g.,

$$x(t) = e^{tA}v = \left( I + tA + \frac{(tA)^2}{2!} + \cdots \right) v$$

$$= v + \lambda tv + \frac{(\lambda t)^2}{2!} v + \cdots$$

$$= e^{\lambda t}v$$

(since $(tA)^k v = (\lambda t)^k v$)
Dynamic interpretation

- for $\lambda \in \mathbb{C}$, solution is complex (we’ll interpret later); for now, assume $\lambda \in \mathbb{R}$
- if initial state is an eigenvector $v$, resulting motion is very simple — always on the line spanned by $v$
- solution $x(t) = e^{\lambda t}v$ is called mode of system $\dot{x} = Ax$ (associated with eigenvalue $\lambda$)
- for $\lambda \in \mathbb{R}$, $\lambda < 0$, mode contracts or shrinks as $t \uparrow$
- for $\lambda \in \mathbb{R}$, $\lambda > 0$, mode expands or grows as $t \uparrow$
Invariant sets

A set $S \subseteq \mathbb{R}^n$ is invariant under $\dot{x} = Ax$ if whenever $x(t) \in S$, then $x(\tau) \in S$ for all $\tau \geq t$

i.e.: once trajectory enters $S$, it stays in $S$

**vector field interpretation:** trajectories only cut into $S$, never out
suppose $Av = \lambda v$, $v \neq 0$, $\lambda \in \mathbb{R}$

- line $\{ tv \mid t \in \mathbb{R} \}$ is invariant
  - (in fact, ray $\{ tv \mid t > 0 \}$ is invariant)
- if $\lambda < 0$, line segment $\{ tv \mid 0 \leq t \leq a \}$ is invariant
Complex eigenvectors

suppose $Av = \lambda v$, $v \neq 0$, $\lambda$ is complex

for $a \in \mathbb{C}$, (complex) trajectory $ae^{\lambda t}v$ satisfies $\dot{x} = Ax$

hence so does (real) trajectory

$$x(t) = \Re \left( ae^{\lambda t}v \right)$$

$$= e^{\sigma t} \begin{bmatrix} v_{\text{re}} & v_{\text{im}} \end{bmatrix} \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

where

$$v = v_{\text{re}} + iv_{\text{im}}, \quad \lambda = \sigma + i\omega, \quad a = \alpha + i\beta$$

- trajectory stays in invariant plane span$\{v_{\text{re}}, v_{\text{im}}\}$
- $\sigma$ gives logarithmic growth/decay factor
- $\omega$ gives angular velocity of rotation in plane
Dynamic interpretation: left eigenvectors

Suppose \( w^T A = \lambda w^T \), \( w \neq 0 \)

Then

\[
\frac{d}{dt} (w^T x) = w^T \dot{x} = w^T Ax = \lambda (w^T x)
\]

i.e., \( w^T x \) satisfies the DE \( \frac{d(w^T x)}{dt} = \lambda (w^T x) \)

Hence \( w^T x(t) = e^{\lambda t} w^T x(0) \)

- Even if trajectory \( x \) is complicated, \( w^T x \) is simple
- If, e.g., \( \lambda \in \mathbb{R}, \lambda < 0 \), halfspace \( \{ z \mid w^T z \leq a \} \) is invariant (for \( a \geq 0 \))
- For \( \lambda = \sigma + i\omega \in \mathbb{C} \), \((\Re w)^T x\) and \((\Im w)^T x\) both have form

\[
e^{\sigma t} (\alpha \cos(\omega t) + \beta \sin(\omega t))
\]
Summary

- *right eigenvectors* are initial conditions from which resulting motion is simple (i.e., remains on line or in plane)

- *left eigenvectors* give linear functions of state that are simple, for any initial condition
Example

\[
\dot{x} = \begin{bmatrix}
-1 & -10 & -10 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} x
\]

block diagram:

![Block Diagram](image)

eigenvalues are $-1$, $\pm i\sqrt{10}$
Example

trajectory with $x(0) = (0, -1, 1)$:
Example

left eigenvector associated with eigenvalue \(-1\) is

\[
g = \begin{bmatrix} 0.1 \\ 0 \\ 1 \end{bmatrix}
\]

let’s check \(g^T x(t)\) when \(x(0) = (0, -1, 1)\) (as above):

![Graph showing the function \(g^T x(t)\) over time]

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Example

eigenvector associated with eigenvalue $i\sqrt{10}$ is

$$v = \begin{bmatrix} -0.554 + i0.771 \\ 0.244 + i0.175 \\ 0.055 - i0.077 \end{bmatrix}$$

so an invariant plane is spanned by

$$v_{\text{re}} = \begin{bmatrix} -0.554 \\ 0.244 \\ 0.055 \end{bmatrix}, \quad v_{\text{im}} = \begin{bmatrix} 0.771 \\ 0.175 \\ -0.077 \end{bmatrix}$$
for example, with $x(0) = v_{re}$ we have
Example: Markov chain

probability distribution satisfies $p(t + 1) = Pp(t)$

$p_i(t) = \text{Prob}( z(t) = i )$ so $\sum_{i=1}^{n} p_i(t) = 1$

$P_{ij} = \text{Prob}( z(t + 1) = i \mid z(t) = j )$, so $\sum_{i=1}^{n} P_{ij} = 1$

(such matrices are called \textit{stochastic})

rewrite as:

$$ [1 \ 1 \ \cdots \ 1] P = [1 \ 1 \ \cdots \ 1] $$

i.e., $[1 \ 1 \ \cdots \ 1]$ is a left eigenvector of $P$ with e.v. 1

hence $\text{det}(I - P) = 0$, so there is a right eigenvector $v \neq 0$ with $Pv = v$

it can be shown that $v$ can be chosen so that $v_i \geq 0$, hence we can normalize $v$ so that $\sum_{i=1}^{n} v_i = 1$

\textbf{interpretation:} $v$ is an \textit{equilibrium distribution}; i.e., if $p(0) = v$ then $p(t) = v$ for all $t \geq 0$

(if $v$ is unique it is called the \textit{steady-state distribution} of the Markov chain)
Modal form

suppose $A$ is diagonalizable by $T$
define new coordinates by $x = T\tilde{x}$, so

$$T\dot{x} = AT\tilde{x} \iff \dot{x} = T^{-1}AT\tilde{x} \iff \dot{x} = \Lambda\tilde{x}$$
Modal form

in new coordinate system, system is diagonal (decoupled):

\[
\int \tilde{x}_1 \\
\int \tilde{x}_n
\]

trajectories consist of \( n \) independent modes, \( i.e., \)

\[
\tilde{x}_i(t) = e^{\lambda_i t} \tilde{x}_i(0)
\]

hence the name *modal form*
Real modal form

when eigenvalues (hence $T$) are complex, system can be put in *real modal form*:

$$S^{-1}AS = \text{diag}(\Lambda_r, M_{r+1}, M_{r+3}, \ldots, M_{n-1})$$

where $\Lambda_r = \text{diag}(\lambda_1, \ldots, \lambda_r)$ are the real eigenvalues, and

$$M_j = \begin{bmatrix} \sigma_j & \omega_j \\ -\omega_j & \sigma_j \end{bmatrix}, \quad \lambda_j = \sigma_j + i\omega_j, \quad j = r + 1, r + 3, \ldots, n$$

where $\lambda_j$ are the complex eigenvalues (one from each conjugate pair)
Real modal form

block diagram of ‘complex mode’:
Diagonalization simplifies many matrix expressions

powers (\textit{i.e.}, discrete-time solution):
\[
A^k = (T \Lambda T^{-1})^k \\
= (T \Lambda T^{-1}) \cdots (T \Lambda T^{-1}) \\
= T \Lambda^k T^{-1} \\
= T \text{diag}(\lambda_1^k, \ldots, \lambda_n^k) T^{-1}
\]

(for \(k < 0\) only if \(A\) invertible, \textit{i.e.}, all \(\lambda_i \neq 0\))
Diagonalization

exponential \( i.e., \ \text{continuous-time solution} \):

\[
e^A = I + A + A^2/2! + \cdots
= I + T\Lambda T^{-1} + (T\Lambda T^{-1})^2 /2! + \cdots
= T(I + \Lambda + \Lambda^2/2! + \cdots)T^{-1}
= Te^\Lambda T^{-1}
= T \text{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})T^{-1}
\]
Analytic function of a matrix

for any analytic function $f : \mathbb{R} \to \mathbb{R}$, i.e., given by power series

$$f(a) = \beta_0 + \beta_1 a + \beta_2 a^2 + \beta_3 a^3 + \cdots$$

we can define $f(A)$ for $A \in \mathbb{R}^{n \times n}$ (i.e., overload $f$) as

$$f(A) = \beta_0 I + \beta_1 A + \beta_2 A^2 + \beta_3 A^3 + \cdots$$

substituting $A = T\Lambda T^{-1}$, we have

$$f(A) = \beta_0 I + \beta_1 A + \beta_2 A^2 + \beta_3 A^3 + \cdots$$
$$= \beta_0 TT^{-1} + \beta_1 T\Lambda T^{-1} + \beta_2 (T\Lambda T^{-1})^2 + \cdots$$
$$= T \left( \beta_0 I + \beta_1 \Lambda + \beta_2 \Lambda^2 + \cdots \right) T^{-1}$$
$$= T \text{diag}(f(\lambda_1), \ldots, f(\lambda_n)) T^{-1}$$
Solution via diagonalization

assume $A$ is diagonalizable

consider LDS $\dot{x} = Ax$, with $T^{-1}AT = \Lambda$

then

$$x(t) = e^{tA}x(0)$$

$$= Te^{\Lambda t}T^{-1}x(0)$$

$$= \sum_{i=1}^{n} e^{\lambda_i t}(w_i^T x(0))v_i$$

thus: any trajectory can be expressed as linear combination of modes
Interpretation

- (left eigenvectors) decompose initial state $x(0)$ into modal components $w_i^T x(0)$
- $e^{\lambda_i t}$ term propagates $i$th mode forward $t$ seconds
- reconstruct state as linear combination of (right) eigenvectors
Qualitative behavior of $x(t)$

- Eigenvalues give exponents that can occur in exponentials.
- Real eigenvalue $\lambda$ corresponds to an exponentially decaying or growing term $e^{\lambda t}$ in solution.
- Complex eigenvalue $\lambda = \sigma + i\omega$ corresponds to decaying or growing sinusoidal term $e^{\sigma t} \cos(\omega t + \phi)$ in solution.
Qualitative behavior of $x(t)$

- $\Re \lambda_j$ gives exponential growth rate (if $> 0$), or exponential decay rate (if $< 0$) of term

- $\Im \lambda_j$ gives frequency of oscillatory term (if $\neq 0$)
Application

for what \(x(0)\) do we have \(x(t) \to 0\) as \(t \to \infty\)?

divide eigenvalues into those with negative real parts

\[ \Re \lambda_1 < 0, \ldots, \Re \lambda_s < 0, \]

and the others,

\[ \Re \lambda_{s+1} \geq 0, \ldots, \Re \lambda_n \geq 0 \]

from

\[ x(t) = \sum_{i=1}^{n} e^{\lambda_i t} (w_i^T x(0)) v_i \]

condition for \(x(t) \to 0\) is:

\[ x(0) \in \text{span}\{v_1, \ldots, v_s\}, \]

or equivalently,

\[ w_i^T x(0) = 0, \quad i = s + 1, \ldots, n \]

(can you prove this?)
Stability of discrete-time systems

suppose $A$ diagonalizable

consider discrete-time LDS $x(t + 1) = Ax(t)$

if $A = T \Lambda T^{-1}$, then $A^k = T \Lambda^k T^{-1}$

then

$$x(t) = A^t x(0) = \sum_{i=1}^{n} \lambda_i^t (w_i^T x(0)) v_i \to 0 \quad \text{as } t \to \infty$$

for all $x(0)$ if and only if

$$|\lambda_i| < 1, \quad i = 1, \ldots, n.$$ 

we will see later that this is true even when $A$ is not diagonalizable, so we have

**fact:** $x(t + 1) = Ax(t)$ is stable if and only if all eigenvalues of $A$ have magnitude less than one